Existence of a Multicameral Core

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Additional Material

(Proofs of Lemmas 1–4 and of Propositions 2 and 6)

Lemma 1 If $x^* f'(p)x$, then $zf'(p)x \forall z \in Co(\{x, x^*\})$.

Proof. The claim follows immediately from the fact that the set of points that each voter prefers to any point is convex. ■

Lemma 2 *For each group and for each direction there exists a unique median hyperplane.*

Proof. (Existence) Note from the definition of a hyperplane and the boundedness of the support of any group that for some hyperplane L:

(i) $\mu(L^+(v,a))$ is (weakly) decreasing in a

(ii) $\mu(L^-(v,a))$ is (weakly) increasing in a

(iii) $\lim_{a \to -\infty} \mu(L^+(v,a)) = 1$

(iv) $\lim_{a \to \infty} \mu(L^-(v,a)) = 1$

Hence from (iii) and (iv) we know that by choosing an arbitrarily high or low value of a we can always ensure either that $\mu(\overline{L^-}) \geq .5$ or $\mu(\overline{L^+}) \geq .5$. Define $a^* = \sup\{a|\mu(\overline{L^+}) \geq .5\}$ and $a_* = \inf\{a|\mu(\overline{L^-}) \geq .5\}$. Monotonicity then implies that $L(v,a)$ is a median hyperplane for $a = \lambda a_* + (1-\lambda)a^*$, $\lambda \in [0,1]$. Hence there is a median hyperplane in every direction.

(Uniqueness) First note that with an odd number of voters, any median hyperplane must contain at least one ideal point. In particular, if $M(v,a)$ is a median hyperplane, then $\mu(M^{\pm}(v,a)) < .5$. Now assume that for some v there are in fact two distinct values, say a_* and a^* such that $M_*(v,a_*)$ and $M^*(v,a^*)$ are each median hyperplanes for a given group. If $a_* < a^*$, then $\overline{M_*^-} \subset M^{*-}$ and hence $\mu(\overline{M_*^-}) \leq \mu(M^{*-})$. Hence $\mu(\overline{M_*^-}) < .5$ and so M_* is not be a median hyperplane. ■

Lemma 3 *If a point, x, does not lie on a median hyperplane of a group then some point on the hyperplane will be preferred by some majority in that group to x.*

Proof. Consider plane $M(v,a)$ median to some group P^j and choose the sign of v, a such that $x \in M^-$. Note that $a - x.v > 0$. Now consider the orthogonal projection of x onto $M(v,a)$, $x^* = x + (a - x.v)v$. We now show that for any point $p_i \in \overline{M^+}$ we have $(p_i - x^*).(p_i - x^*) < (p_i - x).(p_i - x)$ and so $u_i(x^*) > u_i(x)$. Since there is a majority of voters from group P^j with ideals in $\overline{M^+}$ a majority of P^j then prefers x^* to x. To see that if $p_i \in \overline{M^+}$ then $(p_i - x^*).(p_i - x^*) < (p_i - x).(p_i - x)$, substitute for x^*:

$$(p_i - x^*).(p_i - x^*) = (p_i - x).(p_i - x) \\ -(a - x.v)[2(p_i.v - x.v) + (a - x.v)]$$

For $p_i \in \overline{M^+}$, the last term in this expression is positive. To see, this recall that $a - x.v > 0$ and note that $p_i \in \overline{M^+}$ implies $p_i.v - a \geq 0$. Together these imply that $(p_i.v - x.v) > 0$ and so $(p_i - x^*).(p_i - x^*) < (p_i - x).(p_i - x)$ ■

Lemma 4 *For two distinct points x and x', choose $v \in S^{n-1}$ and $\varepsilon > 0$, such that $x' = x + \varepsilon v$. If a majority of group j prefers x' to x then $M_v^j \subseteq L^+(v, x.v)$ and $\mu^j(L^+(v, x.v)) > .5$.*

Proof. Note first by the definition of median planes that $M_v^j \subseteq L^+(v, x.v)$ *if and only if* $\mu^j(L^+(v, x.v)) > .5$. Now assume contrary to the claim that $\mu^j(L^-(v, x.v)) \geq .5$. and in particular (since there is an odd number of voters in each group) $\mu^j(\overline{L^-}(v, x.v)) > .5$. Writing the median plane for group j with directional vector v as $M^j(v, a)$, we then have that $a < x.v$. In this case, and, from Lemmas 1 and 3, x (a convex combination of x' and the orthogonal projection of x' onto M_v^j) is preferred by a majority of group j voters to x'. ■

Proposition 2: The core is convex. Proof. (By Contradiction) Consider two points in the core, q and q' and a strictly convex combination of them, $z = \lambda q + (1 - \lambda)q'$, $\lambda \in (0, 1)$ that is not in the core. If z is not in the core then there exists a point $z^* \neq z$ that a majority in all groups prefer to z. Define $v = \frac{z^* - z}{||z^* - z||}$ and consider the hyperplane $L(v, z.v)$. Note that by construction, z lies on $L(v, z.v)$ and z^* lies in $L^+(v, z.v)$. Assume, without loss of generality, that $q.v \leq q'.v$. Then, since $z.v = [\lambda q + (1 - \lambda)q'].v = \lambda q.v + (1 - \lambda)q'.v$, we have $q.v \leq z.v \leq q'.v$.

Consider arbitrary group j. Since a majority of voters in group j prefers z^* to z, Lemma 4 implies that median hyperplane M_v^j lies strictly within the upper half-space $L^+(v, v.z)$ and hence a majority of points in P^j lies within $L^+(v, z.v)$. $(*)$

But, since q lies in the core, for some group, j, a minority of the points in P^j lies in $L^+(v, q.v)$ (Since otherwise, from Lemmas 1 and 3, some point on $L(v, q.v)$ would be preferred by majorities in all groups to q). Since $q.v \leq z.v$, we have $L^+(v, z.v) \subseteq L^+(v, q.v)$ and so a minority of group j lies within $L^+(v, z.v)$. $(**)$

Since $(*)$ contradicts $(**)$ we have that the core is convex. ■

Proposition 6 *Assume $m \geq 2$ and $n > \frac{4m-2}{3}$. If ideal points are in general position, then the core is empty.*

The strategy of proof is to show that under the conditions of the propositions not all ham sandwich cuts intersect. The proof makes use of the following lemma (illustrated in Figure 1) in order to identify regions where planes intersect.

Lemma 5 *Consider two sets of points in general position A_1 and A_2 in R^n, with $|A_1 \cup A_2| < n+2$. If for some point x^*, $x^* \in$Aff$(A_1) \cap$Aff(A_2) then $x^* \in$Aff$(A_1 \cap A_2)$.*

Proof. Consider two sets of points in general position $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{R}^n$, with $|\mathcal{A}_1 \cup \mathcal{A}_2| < n+2$. If for some point x^*, $x^* \in \text{Aff}(\mathcal{A}_1) \cap \text{Aff}(\mathcal{A}_2)$ then $x^* \in \text{Aff}(\mathcal{A}_1 \cap \mathcal{A}_2)$. We may write $x^* = \sum_{i=1}^{|\mathcal{A}_1|} \lambda_1^i q_1^i = \sum_{i=1}^{|\mathcal{A}_2|} \lambda_2^i q_2^i$ for $q_j^1, q_j^2, ..., q_j^{|\mathcal{A}_j|} \in \mathcal{A}_j$ and coefficients $\{\lambda_j^i\}_{i=1,2,...,|\mathcal{A}_j|}$ such that $\sum_{i=1}^{|\mathcal{A}_j|} \lambda_j^i = 1$. Assume, contrary to the proposition that for some arbitrary point $q_j^i \in \mathcal{A}_j \backslash (\mathcal{A}_1 \cap \mathcal{A}_2)$, we have $\lambda_j^i \neq 0$. For concreteness assume $q_1^1 \notin \mathcal{A}_2$ and $\lambda_1^1 \neq 0$. In this case we may write $q_1^1 = \sum_{i=1}^{|\mathcal{A}_2|} \frac{\lambda_2^i}{\lambda_1^1} q_2^i - \sum_{i=2}^{|\mathcal{A}_1|} \frac{\lambda_1^i}{\lambda_1^1} q_1^i$. Note however that $\sum_{i=1}^{|\mathcal{A}_2|} \frac{\lambda_2^i}{\lambda_1^1} - \sum_{i=2}^{|\mathcal{A}_1|} \frac{\lambda_1^i}{\lambda_1^1} = \frac{1}{\lambda_1^1} - \frac{1-\lambda_1^1}{\lambda_1^1} = 1$ and hence q_1^1 lies in the affine hull of $(\mathcal{A}_1 \cup \mathcal{A}_2) \backslash q^1$. Defining $k = |\mathcal{A}_1 \cup \mathcal{A}_2|$ we now have that k points lie on a $(k-2)$-hyperplane, a possibility that is ruled out for points in general position whenever $k < n+2$. ■

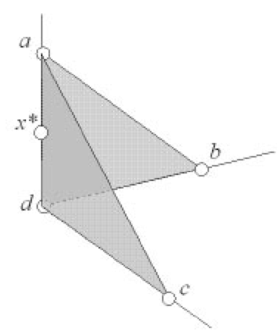

Figure 1: Illustration of Lemma 5. The point x^* lies on Aff$(a,d,b) \cap$ Aff(a,d,c) and so it lies on Aff$((a,d,b) \cap (a,d,c))$ =Aff(a,d).

Proof. (of Proposition) We fix $p \in \mathcal{P}$ and assume that the points in p are in general position. The key step is to show that with $\frac{4m-2}{3} < n \leq \frac{3}{2}m$ the intersection of all ham sandwich cuts through $P(p)$ is empty. By considering projections onto lower dimensional subspaces we see that if no x^* lies on all cuts when $n \leq \frac{3}{2}m$ then no x^* lies on all cuts when $n > \frac{3}{2}m$. Hence, because the core is contained in the intersection of all ham sandwich cuts (from Lemma 3), it is empty.

We assume that a point x^* is in the core. We then construct special ham sandwich cuts whose intersection is empty. First, we need the following claim. ■

Claim 1 *Consider any partitioning of $P(p)$ into $\{P_1, P_2\}$ with $|P_1| = n - m$ and $|P_2| = 2m - n$. There exists a ham sandwich cut, H, through P that contains exactly 2 ideal points from each group in P_1 and exactly one point from each group in P_2.*

Proof. (of claim) Partition P into $\{P_1, P_2\}$ with $|P_1| = n - m$ and $|P_2| = 2m - n$. Given an arbitrary pair of ideal points $\{p_1^k, p_2^k\}$ from each collection $P^k \in P_1$ we consider the collection:

$\tilde{P} = \{\{P^k \backslash \{p_1^k, p_2^k\}\}_{k=1,2,\ldots,|P_1|}, \{p_1^k\}_{k=1,2,\ldots,|P_1|}, \{P^k\}_{k=1,2,\ldots,|P_2|}\}$.

Since $\tilde{P}$ contains $2|P_1| + |P_2| = n$ sets we have from the ham sandwich theorem that some ham sandwich cut H through $\tilde{P}$ exists. Let v denote the directional vector of H. Since each set in $\tilde{P}$ has an odd number of points, H contains at least one point from each of these sets. However with ideal points in general position, H, an $(n-1)$-dimensional hyperplane, may contain at most n ideal points. Hence H contains exactly one point from each of the n groups in $\tilde{P}$, and consequently exactly 2 points from each group in P_1 and exactly one point from each group in P_2. It remains to show that H is a ham sandwich cut through P as well. For each group P^k in P_1, because H is median to $P^k \backslash \{p_1^k, p_2^k\}$ and contains exactly one point p_3^k from $P^k \backslash \{p_1^k, p_2^k\}$, each open half-space of H contains $\frac{|P^k|-3}{2}$ points from $P^k \backslash \{p_1^k, p_2^k\}$. Since H also contains p_1^k and p_3^k, each open half-space of H contains at most $\frac{|P^k|-1}{2} < \frac{|P^k|}{2}$ points from P^k. Hence H is median to each P^k in P_1. By construction, H is also median to each P^h in P_2. Thus, H is a ham sandwich cut through P that contains exactly 2 ideal points from each group in P_1 and exactly one point from each group in P_2. ■

The next step is to show that there exists a collection of ham sandwich cuts that intersect on the affine hull of points in P_2. This is established in the next claim.

Claim 2 *Consider any partitioning of P into $\{P_1, P_2\}$ with $|P_1| = n - m$ and $|P_2| = 2m - n$. Let H denote a ham sandwich cut, through P that contains exactly 2 ideal points from each group in P_1 and exactly one point from each group in P_2. Let $\mathcal{A}$ denote the set of $2m - n$ points from groups in $|P_2|$ that lie on H. Then $x^* \in$Aff($\mathcal{A}$).*

Proof. (of claim) Let $\mathcal{A}$ denote the collection of points q_1^k from P_2 lying on the hyperplane H. Because the ideal points are in general position, the ham sandwich cut H may be written H=Aff$(\{p_1^k, p_3^k\}_{k=1,2,\ldots,|P_1|}, \mathcal{A})$

and so $x^* \in$Aff$(\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|}, \mathcal{A})$. We now sequentially delete each element p_j^k from the set $(\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|}, \mathcal{A})$. To delete p_1^1 choose ε small enough (with $\varepsilon > 0$ if $p_2^1 \in H^+$ and $\varepsilon < 0$ if $p_2^1 \in H^-$) such that $\tilde{H} =$Aff$((\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash p_1^1, p_1^1 + \varepsilon v, \mathcal{A})$ is a ham sandwich cut. From Lemma 5, we then have that since x^* lies on H and on $\tilde{H}$ that $x^* \in$Aff$((\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash p_1^1, \mathcal{A})$. Repeating this operation for p_3^1 we have $x^* \in$Aff$((\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash p_3^1, \mathcal{A})$. Applying Lemma 5 to sets $(\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash p_1^1, \mathcal{A})$ and $(\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash p_3^1, \mathcal{A})$ we then have $x^* \in$Aff$((\{p_1^k, p_3^k\}_{k=1,2,...,|P_1|})\backslash \{p_1^1, p_3^1\}, \mathcal{A})$. Continuing in this manner for $k = 2, 3, ..., |P_1|$ we have $x^* \in$Aff$(\varnothing, \mathcal{A}) =Aff(\mathcal{A})$. ■

Proof. (of Proposition) The final step is to repeat this argument for different partitionings of P.

First consider a second partitioning $\{P_1', P_2'\}$. with $P_1 \subseteq P_2'$. (such a partitioning is possible since with $n \leq \frac{3}{2}m$ we have $n - m \leq 2m - n$). In this case P_2 and P_2' contain points drawn from at most $3m - 2n$ of the same groups.

Reapplying the preceding claims we have $x^* \in$Aff$(\mathcal{A}')$, where $\mathcal{A}'$ denotes the collection of points in P_2' that lie on the corresponding ham sandwich cut H'.

But then since P_2 and P_2' share at most $3m - 2n$ groups, the set $\mathcal{B} = \mathcal{A} \cap \mathcal{A}'$, contains one point from each of at most $3m - 2n$ different groups. Now, letting k denote the number of distinct points in $\mathcal{A} \cup \mathcal{A}'$, we have that $k \leq 2(2m - n)$. Additionally, since $n > \frac{4m-2}{3}$ we have that $2(2m - n) < n + 2$ and hence $k < n + 2$. With $x^* \in$Aff$(\mathcal{A})$, $x^* \in$Aff$(\mathcal{A}')$ and $|\mathcal{A} \cup \mathcal{A}'| < n + 2$, Lemma 5 then implies that $x^* \in$Aff$(\mathcal{B})$. If $\mathcal{B} = \varnothing$ then we are done.

If $\mathcal{B} \neq \varnothing$, we need to construct $\mathcal{B}'$ in the same manner such that $x^* \in$Aff$(\mathcal{B} \cap \mathcal{B}')$ and $\mathcal{B} \cap \mathcal{B}' = \varnothing$. To this end, recall that $\mathcal{B}$ may be selected to contain one point from each of at most $3m - 2n$ arbitrary groups. If $m \geq 2(3m - 2n)$, or equivalently, if $n \geq \frac{5}{4}m$, we can ensure that $\mathcal{B} \cap \mathcal{B}' = \varnothing$ by constraining $\mathcal{B}'$ to be drawn from a set of $3m - 2n$ groups whose elements do not appear in $\mathcal{B}$. Because $n \geq \frac{4m+1}{3} \geq \frac{5}{4}m$ the condition always holds and we can select a $\mathcal{B}'$ such that $\mathcal{B} \cap \mathcal{B}' = \varnothing$.

Lastly, it is straightforward to check that $k' = |\mathcal{B} \cup \mathcal{B}'|$ satisfies the conditions of Lemma 5 and hence that $x^* \in$Aff$(\mathcal{B} \cap \mathcal{B}') = \varnothing$, proving the proposition. ■