To Bargain or to Brawl? Politics in Institutionally Weak Environments

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Abstract

This paper considers political interactions between internally fragmented groups in institutionally weak environments. In these environments, political actors cannot depend on majority decisions to be enforced or expect bargained agreements to be honored. Instead, they try to unilaterally alter the status quo, using violence or influence to pull it one way or another.

A status quo in this context is considered stable when the combined effects of the equilibrium strategies of different actors serve to cancel each other out exactly – leaving the status quo untouched. Ironically, the –costly– equilibrium of the game turns out to have equivalence properties with solutions in institutionally rich environments; in these environments the same outcome would be achieved without the costs incurred in institutionally weak environments.

The aim of the project is to address questions of the form: how do outcomes in institutionally weak environments compare to outcomes in institutionally rich environments? how does the aggregate level of violence vary with the coalitional, or perhaps ethnolinguistic structure of a polity? how does it vary as a function of the degree of homogeneity within organized groups and the heterogeneity across groups? or with the cost of violence? and how do political outcomes vary as the distribution of preferences become more polarized?

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1 Introduction

> The theoreticians of the Popular Front do not essentially go beyond the first rule of arithmetic, that is, addition: "Communists" plus Socialists plus Anarchists plus Liberals add up to a total which is greater than their respective isolated numbers. Such is all their wisdom. However, arithmetic alone does not suffice here. One needs as well at least mechanics. The law of the parallelogram of forces applies to politics as well. In such a parallelogram, we know that the resultant is shorter, the more component forces diverge from each other. When political allies tend to pull in opposite directions, the resultant proves equal to zero.
>
> *Leon Trotsky*
>
> "The Lessons of Spain: The Last Warning," *Socialist Appeal* January 8th and 15th, 1938

This paper mixes non-cooperative and cooperative game theory to study political action by internally fragmented groups in institutionally weak environments. Unlike many models of political decision making, I do not assume that political actors may write enforceable contracts with each other or that decisions voted or agreed upon will be implemented. I allow for the possibility however that such cooperative action can be achieved within subgroups of a population –within groups that, for whatever reason, have overcome collective action problems. In a Hobbesian world, such subgroups consist of single persons. Groups that fail to solve collective action problems impose externalities upon each other. They take actions unilaterally to push or pull state variables one way or another in ways that may greatly harm other actors.

To model these processes I adopt the framework of the spatial model, a framework typically used for modelling cooperative games. By using a spatial framework to study non-cooperative games between groups we can allow for very rich policy spaces within which players act. And we can assume arbitrary distributions of preferences over these spaces. It could for example be that the policy space represents a set of policies being put in place in a one party state, it could represent the distribution of land seized by warlords in Liberia or Somalia, it could represent the level of effective taxation on different commodities. And so on. In general we may expect that the status quo does not satisfy *all* players and that sometimes joint gains may be made by some or all players whereas at other times improvements for some imply worse situations for others. It could be that players are clustered by interest, perhaps in accordance with their ethnic group or class status. Or cross-cutting cleavages may exist, with members of identifiable groups spread throughout the policy space.

In all cases I assume that although institutions may exist that require that players cooperate, at any point in time players may be able to ignore the institutions and force a change in the status quo. They could take military action to attempt to seize resources from one or another player, or place diplomatic or not so diplomatic pressure on policy makers, bureaucrats or other political actors. If players take such unilateral actions we may expect some change in the status quo. But any new status quo reached through such unilateral actions may in turn be susceptible to further attempts by players to alter it. A stable state in this context would be one where actions by players leave the status quo unchanged, given that the action by each player is a best response to the actions of the others, at every point in time.

The research project has three components. The first component addresses situations where no bargains are possible and all players act unilaterally. The second section produces results when coalitional structures exist and are given exogenously. In this context a "coalitional structure" is a partitioning of the population of a polity into groups within which efficient bargaining is possible. These "coalitions" are subgroups that for whatever reasons have succeeded in providing institutions to enforce contracts among their members. If we believe that ethnolinguistic groups have an ability to enforce deals among their members then it is natural to interpret the coalitional structure of a polity in terms of the ethnolinguistic fragmentation of the country. These first two parts are presented in this paper. The third part, underway as a companion paper, uses these results to endogenize the coalitional structure of a polity. It addresses the questions: when will players choose to bargain and when will they choose to brawl? Is there a relation between a player's preferences relative to those of the rest of the group and his incentives to bargain or to brawl?

This paper lays the foundations for this work. And it provides a structure in which we can address questions of the form: how do outcomes in institutionally weak environments compare to outcomes in institutionally rich environments? how does the aggregate level of violence vary with the coalitional, or perhaps ethnolinguistic structure of a polity? how does the aggregate level of expenditure on violent activity vary with the cost of violence? with the distribution of strength in a polity changes? or with the degree of homogeneity within organized groups and the heterogeneity across groups?

2 The Model

Action takes place in discrete time. I use subscripts when necessary to indicate periods of play. Subscripts are also used to index dimensions and superscripts are used to index players or to identify groups. Sets, such as the set of all players, $\mathcal{M}$, a set of players in some coalition, $\mathcal{C}^k$, or some set of coalitions, $\mathcal{C}$, are denoted by capital letters in script. I use $|\mathcal{A}|$ to denote the number of elements in a set, $\mathcal{A}$.

At the beginning of any period of play the state over which rivalry exists is described by a vector $y_{t-1} \in \mathbb{R}^n$. At every moment in time, players each optimally choose a level and direction of force to apply to this state. Formally, we say that an "**action**" in time t for a player i is a vector, v_t^i drawn from an action space, $\mathbb{R}^n$. To provide a mapping from the possibly conflicting actions of the players to changes in the state variable, I assume that changes in the state satisfy Newton's laws of motion, that is, that the net force acting upon a point is determined by computing the vector sum of all the individual forces acting upon the point.[1] Hence we have $y_t = y_{t-1} + \sum\limits_{i \in \mathcal{M}} v_t^i$.

A "**strategy**" for a player in time t, σ_t^i consists of a collection of actions $\sigma_t^i = \{v_{t+s-1}^i\}_{s=1,2,..T}$ for games of length T beginning in period t. In a one period game, $\sigma_t^i = v_t^i$. A **Nash equilibrium** profile of strategies in some period of play, $\sigma_t^* \equiv \{\sigma_t^{i*}\}_{i \in \mathcal{M}}$ is a profile with the property that each strategy by a player, i, σ_t^{i*}, maximizes her utility given the strategies of all the other players, $\sigma_t / \{\sigma_t^{i*}\}$. I refer to elements of such a profile as "**equilibrium strategies.**" Equilibrium strategies have typical elements v_t^{i*}, which I refer to as "**equilibrium actions.**" I say that a point, y^*, is an "**equilibrium point**" or a "**stable point**" if at time t, $y^* + \sum\limits_{s=1}^{r} \sum\limits_{i \in \mathcal{M}} v_{t+s-1}^{i*} = y^*$ for $r = 1...T$. That is, a point is stable if it remains fixed through all subsequent rounds when all players play equilibrium strategies; or alternatively, if all equilibrium actions by all players exactly offset each other in all subsequent rounds. The "**magnitude**" of force exerted by applying a vector v_t^i to the status quo is given by the length of v_t^i, $|v_t^i|$.

I assume that players pay a **cost**, $c^i(v_t^i)$ of applying a force, v_t^i, and

[1] More complex mappings or families of mappings could also be considered. It is striking however how closely the Newtonian mapping informs more informal treatments of conflictual political action, such as Trotsky's description of action among allies in Spain. Before Trotsky, Engels wrote: "History is made in such a way that the final result always arises from conflicts between many individual wills, of which each again has been made what it is by a host of particular conditions of life. Thus there are innumerable intersecting forces, an infinite series of parallelograms of forces which give rise to one resultant: the historical event." letter to Bloch Sept 21 1890.

that this cost is quadratic in the magnitude of the force applied. I allow different actors to vary in the costs they face for any given magnitude of force applied: $c^i(v^i) = \frac{|v^i|^2}{\pi^i}$, where π^i, $0 < \pi^i < \infty$, a scalar, is a measure of **strength** (the greater the strength, the lower the costs of any given action).[2] I assume that **utility functions** take an additive form with constant discount rates: $V_t^i = u^i(y_t) - c^i(v_t^i) + \delta\left[u^i(y_{t+1}) - c^i(v_{t+1}^i)\right] + ... + \delta^{T-1}\left[u^i(y_T) - c^i(v_T^i)\right]$. Hence, utility is additive in utility in the state variable, y_t, and in costs and it is also additive over time. For developing most results I also assume that the **subutility function** $u^i(y_t)$ is quadratic in distance from a player-specific, time invariant ideal point: $x^i \in \mathbb{R}^n$, $u^i(y_t) = -(|y_t - x^i|)^2$.

A "**coalition**" is a set of players drawn from $\mathcal{M}$. A "**coalitional structure**" is a partitioning of $\mathcal{M}$ into a set of coalitions.[3] The "**grand coalitional structure**," denoted by $\{\mathcal{M}\}$, is the coalition that includes all players, $\{1, 2, 3...|\mathcal{M}|\}$. The "**atomistic coalitional structure**," denoted by $\langle\mathcal{M}\rangle$, is the coalitional structure where each coalition consists of a single player, $\{\{1\}, \{2\}, \{3\}...\{|\mathcal{M}|\}\}$. The partitioning of any subset, $\mathcal{A}$, of $\mathcal{M}$ into singletons is similarly denoted $\langle\mathcal{A}\rangle$. The "**dispersion**" of a group, $\mathcal{A}$, is written $\rho(\mathcal{A})$ and is defined for players, $i \in \mathcal{A}$, with ideal points in n dimensions as the sum over all the dimensions of the variance of the distribution of the ideals of the players on each dimension : $\rho(\mathcal{A}) = \frac{1}{|\mathcal{A}|}\sum_{j=1}^{n}\sum_{i\in\mathcal{A}}\left[x_j^i - \bar{x}_j\right]^2$. The "**weighted dispersion**" of a group $\mathcal{A}$, given some weighting vector $\pi \in \mathbb{R}^{|\mathcal{A}|}$ is given by $\rho(\mathcal{A}|\pi) \equiv \frac{1}{\sum_{i\in\mathcal{A}}\pi^i}\sum_{j=1}^{n}\sum_{i\in\mathcal{A}}\pi^i\left[x_j^i - \frac{\sum_{i\in\mathcal{A}}\pi^i x^i}{\sum_{i\in\mathcal{A}}\pi^i}\right]^2$. That, is $\rho(\mathcal{A}|\pi)$ is the sum of the weighted variance of the group over each dimension. Note that if π describes uniform weights, then $\rho(\mathcal{A}|\pi) = \rho(\mathcal{A})$. Note also that $\rho(\mathcal{A}|\pi)$ is homogenous of degree 0 in π**.** Finally, I say that a player, $i \in \mathcal{A}$, is a "**dominant player**" with respect to a group, $\mathcal{A}$, if $\pi^i \geq \frac{1}{2}\sum_{k\in\mathcal{A}}\pi^k$.

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[2] π^i could also be interpreted as the reciprocal of the *price* that players face to acquire military technology. Under this interpretation a quantity of technology $|v^i|^2$, sold to i at the unit price $\frac{1}{\pi^i}$ is required to purchase an action of magnitude $|v^i|$.

[3] That $\mathcal{C}$ is a *partitioning* implies that if $\mathcal{C}^1$, $\mathcal{C}^2$,... $\mathcal{C}^{|\mathcal{C}|}$ are coalitions, and $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2, ...\mathcal{C}^{|\mathcal{C}|}\}$ is a coalitional structure then (1) $\mathcal{C}^k \cap \mathcal{C}^h = 0$ for any $\mathcal{C}^k \neq \mathcal{C}^h$ in $\mathcal{C}$ and (2) $\bigcup_{\mathcal{C}^k\in\mathcal{C}}\mathcal{C}^k = \mathcal{M}$.

3 The Atomistic Game: The War of All Against All

> Me and my clan against the world;
> Me and my family against my clan;
> Me and my brother against my family;
> Me against my brother.
> - *The hierarchy of priorities, as ordered by a Somali proverb* (Peterson 2000)

In this section I consider the game where there are no coalitions. Each actor acts atomistically attempting to maximize his utility while imposing externalities on other players. I first consider the stage game where all players have quadratic subutility functions. The stage game is important insofar as it characterizes situations where actions may only be taken once or situations where actions may be taken multiple times but players have extremely high discount rates. I then turn to the iterated stage game, asking how policy changes when myopic players are allowed to repeat the stage game for an arbitrarily large but finite number of periods. I then turn to the situation where players are sophisticated and take account of equilibrium actions that they and other players will take in the future. It turns out that when players are of equal strength, the stable point is the same in the stage game, in the repeated game with myopic players and in the repeated game with sophisticated players. In all cases the stable point is given by the mean of the ideal points of the players. For the utility functions assumed, this point is also the utilitarian welfare maximizing point. When players are asymmetric in strength, the stable point in repeated game with sophisticated players differs from that of the stage game and the repeated game with myopic players, tending, *ceteris paribus*, more towards the ideals of the stronger players. These key results are developed under the assumption of quadratic utility. In the fourth part of this section, however, I consider a very broad class of utility functions and demonstrate that for this class of utility functions a point is stable if and only if it is the weighted utilitarian maximum, where weights are given by the relative strengths of the players. I end with a short section that summarizes the key finding of this section and that provides an example of equilibrium actions in a two person game in two dimensions in order to elucidate the key results from these sections.

3.1 The Stage Game

The simplest version of the model assumes that players' preferences over policy are quadratic and that players consider the impacts of their actions for a single period of play at a time. This last assumption may hold if players believe that the game is only to be played once or if players have very high discount rates.

The two assumptions are captured by the simple utility function $V_t^i = -(|y_t - x^i|)^2 - \frac{|v_t^i|^2}{\pi^i}$.

The timing of the game is as follows. At the beginning of period t, the status quo is fixed at y_t. Players in $\mathcal{M}$ then optimally choose a set of vectors $\{v_t^i\}_{i \in \mathcal{M}}$. At the end of the period, the new policy option is given by $y_t = y_{t-1} + \sum\limits_{i \in \mathcal{M}} v_t^i$ and players enjoy utility $V_t^i(y_t, v_t^i)$.

I solve the problem for $|\mathcal{M}|$ players in n dimensions. Note first that each player chooses an n-dimensional vector v_t^{i*}such that:

$$v_t^{i*} = \arg\max_{v_t^i}\{-\sum_{j=1}^{n}(x_j^i - (y_{t-1,j} + v_{t,j}^1 + \ldots + v_{t,j}^i + \ldots + v_{t,j}^{|\mathcal{M}|}))^2 - \frac{|v_t^i|^2}{\pi^i}\} \quad (1)$$

For player i the first order conditions for an arbitrary component of v^{i*}, say $v_{t,j}^{i*}$ is:

$$2(x_j^i - y_{t-1,j} - v_{t,j}^1 - v_{t,j}^2 \ldots - v_{t,j}^{i*} \ldots - v_{t,j}^{|\mathcal{M}|}) - \frac{2v_{t,j}^{i*}}{\pi^i} = 0 \quad (2)$$

or

$$v_{t,j}^{i*} = \frac{\pi^i}{\pi^i + 1} x_j^i - \frac{\pi^i}{\pi^i + 1}\left(y_{t-1,j} + \sum_{k \in \mathcal{M}/\{i\}} v_{t,j}^k\right) \quad (3)$$

And similar conditions exist for all other players $i \in \mathcal{M}$ and dimensions $j \in \{1, 2 \ldots n\}$. In equilibrium, all equations are satisfied by the set $\{v_{t,j}^i = v_{t,j}^{i*} | i \in \mathcal{M},\ \ j \in \{1, 2 \ldots n\}\}$. We have then for each dimension, $|\mathcal{M}|$ independent equations in $|\mathcal{M}|$ unknowns. Hence, we can solve for v_t^{i*} for all i on any dimension j.

Solving the equations in groups of $|\mathcal{M}|$ for an arbitrary dimension, j, we get for each i:

$$v^{i*}(y_{t-1}) = \left[x^i - \frac{y_{t-1} + \sum\limits_{k \in \mathcal{M}} \pi^k x^k}{1 + \sum\limits_{k \in \mathcal{M}} \pi^k}\right]\pi^i \quad (4)$$

The policy that results from these actions is given by

$$y_t = y_{t-1} + \sum_{h \in \mathcal{M}} \left[x^h - \frac{y_{t-1} + \sum\limits_{k \in \mathcal{M}} \pi^k x^k}{1 + \sum\limits_{k \in \mathcal{M}} \pi^k} \right] \pi^h = \frac{1}{1 + \sum\limits_{k \in \mathcal{M}} \pi^k} (y_{t-1} + \sum_{k \in \mathcal{M}} \pi^k x^k) \tag{5}$$

I now show that an equilibrium point exists and is unique. More constructively, I show that the value taken by the equilibrium point is the weighted average of the ideal points of all players, where the weights are given by the strengths of each player. Strong players will normally find the equilibrium close to their ideals and weak players will find it far away. The weighted average is of course internal to the Pareto set; [4] it is also the point that maximizes a weighted utilitarian social welfare function where weights are given by the strengths of the players.

Proposition 1 *A status quo policy in the stage game with quadratic utility and costs is an equilibrium point if and only if it is the weighted average of the ideals of the players in $\mathcal{M}$ where the weights are given by the relative strengths of the players. That is y is an equilibrium point if and only if* $y = \frac{\sum_{i \in \mathcal{M}} \pi^i x^i}{\sum_{i \in \mathcal{M}} \pi^i}$.

Proof. A steady state in the basic model is a status quo policy y^* that induces responses $\{v^{i*}(y^*)\}_{i \in \mathcal{M}}$ such that $\sum\limits_{i \in \mathcal{M}} v^{i*}(y^*) = 0$. This value may be easily calculated by summing Equation 3 over all i in $\mathcal{M}$ and solving for $\sum\limits_{i \in \mathcal{M}} v_j^{i*}(y^*)$. This gives:

$$\sum_{i \in \mathcal{M}} v_j^{i*}(y^*) = \frac{\sum\limits_{i \in \mathcal{M}} \pi^i}{1 + \sum\limits_{i \in \mathcal{M}} \pi^i} (x_j^i - y_j^*) \tag{6}$$

which vanishes if and only if $\sum\limits_{i \in \mathcal{M}} \pi^i (x_j^i - y_j^*) = 0$ or (in vector form):

$$y^* = \frac{\sum\limits_{i \in \mathcal{M}} \pi^i x^i}{\sum\limits_{i \in \mathcal{M}} \pi^i} \tag{7}$$

■

[4] Since the weighted average is a convex combination of ideals and the Pareto set (given our assumptions on the subutility function) is the set of all convex combinations of ideals.

We now have the information we need to enquire into the level of force expended in equilibrium. Using Equation 4 we have that the force applied by each player along each dimension at the equilibrium point is given by:

$$v_j^{i*} = \left[x_j^i - \frac{\frac{\sum_{k\in\mathcal{M}} \pi^k x_j^k}{\sum_{k\in\mathcal{M}} \pi^k} + \sum_{k\in\mathcal{M}} \pi^k x_j^k}{1 + \sum_{k\in\mathcal{M}} \pi^k} \right] \pi^i \tag{8}$$

This simplifies easily to:

$$v^{i*} = \left[x^i - y^* \right] \pi^i \tag{9}$$

The interpretation of Equation 9 is that in equilibrium, players' strategies are "truthful" in the sense that they try to pull policy directly towards their ideal. If it so happens for example that the equilibrium is the ideal of some player, then that player will not have to expend any energy to maintain the equilibrium.[5] If policy is in equilibrium then, the equilibrium strategies are preference revealing.

The costs born by players in equilibrium is then:

$$c^i(v^{i*}) = |x^i - y^*|^2 \pi^i \tag{10}$$

The following proposition uses Equation 10 to show that the average deadweight losses (or the average amount of violence employed) resulting from optimal actions taken when policy is at the equilibrium point are increasing in the average strength of the players (that is, the cheaper action is on average, the more will be spent in the aggregate). Furthermore, the average (and total deadweight losses) are directly proportional to the strength of the players.

Proposition 2 *The total cost paid by players when policy is at the equilibrium point is directly proportional to the degree of (weighted) dispersion of the group. Furthermore it is increasing with proportionate increases in the strength of the players.*

Proof. Summing Equation 10 over the individuals we find

$$\sum_{i\in\mathcal{M}} c^i(v^{i*}) = \sum_{i\in\mathcal{M}} \sum_{j=1}^{n} \pi^i \left[x_j^i - \frac{\sum_{i\in\mathcal{M}} \pi^i x^i}{\sum_{i\in\mathcal{M}} \pi^i} \right]^2 = \rho(\mathcal{M}|\boldsymbol{\pi}) \sum_{i\in\mathcal{M}} \pi^i \tag{11}$$

[5] This could be the happy situation of a weak player centered between two strong players.

This establishes the first part of the proposition. The second part follows from the fact that $\rho(\mathcal{A}|\boldsymbol{\pi})$ is homogenous of degree 0 in $\boldsymbol{\pi}$ and so, for some scalar α, $\partial\rho(\mathcal{M}|\alpha\boldsymbol{\pi})\sum_{i\in\mathcal{M}}\alpha\pi^i/\partial\alpha = \sum_{i\in\mathcal{M}}\pi^i\rho(\mathcal{M}|\boldsymbol{\pi})$. ■

I now turn to consider the comparative statics of effort as a function of the strength of individual players. I ask whether the amount of violence employed and the total costs incurred by an individual player in equilibrium are increasing or decreasing in the strength of a player. That is, if violence is cheaper will individuals purchase more of it? And will they spend larger or smaller shares of their income on it? Results of the inquiry are summarized in the next proposition.

Proposition 3 *In equilibrium, stronger players employ a greater amount of force than weaker players, ceteris paribus. The total amount that a player pays in equilibrium is decreasing in the cost of the action if and only if a player is a dominant player.*

Proof. From Equation 7 and Equation 9 we have that

$$\frac{\partial v_j^{i*}}{\partial\pi^i} = (x_j^i - y_j^*)\frac{\sum_{k\in M}\pi^k - \pi^i}{\sum_{k\in M}\pi^k} \tag{12}$$

$$\frac{\partial|v^{i*}|^2}{\partial\pi^i} = 2\sum_{j=1}^{n}\pi^i(x_j^i - y_j^*)^2\frac{\sum_{k\in M}\pi^k - \pi^i}{\sum_{k\in M}\pi^k} = 2\pi^i\frac{\sum_{k\in M}\pi^k - \pi^i}{\sum_{k\in M}\pi^k}|x^i - y^*|^2 > 0 \tag{13}$$

Equation 12 states that if, $x_j^i > y_j^*$ then $\frac{\partial v_j^{i*}}{\partial\pi^i} > 0$, that is if a player stands to one side of the equilibrium point, then the stronger she is, the more she shall pull to that side. Equation 13 confirms that the (square of) the magnitude of force applied increases with strength (and hence the magnitude also increases with strength). This establishes the first part of the proposition. For the second part of the proposition, note that:

$$\frac{\partial\frac{|v^{i*}|^2}{\pi^i}}{\partial\pi^i} = \partial\left[|x^i - \frac{\sum_{k\in\mathcal{M}/\{i\}}\pi^k x^k + \pi^i x^i}{\sum_{k\in\mathcal{M}/\{i\}}\pi^k + \pi^i}|^2\pi^i\right]/\partial\pi^i = \frac{\sum_{k\in\mathcal{M}}\pi^k - 2\pi^i}{\sum_{k\in\mathcal{M}}\pi^k}|x^i - y^*|^2 \tag{14}$$

The interpretation of Equation 14 is that the amount paid in effort by a player at the status quo is increasing in her strength if and only if $\pi^i < \frac{1}{2} \sum\limits_{k \in \mathcal{M}} \pi^k$. That is, the gross amount that the player pays is *decreasing* in the cost of the action if and only if a player is a *dominant* player. For non-dominant players, the cheaper the action (the stronger they are) the more they pay, *ceteris paribus.* ■

The second part of the proposition implies that in 2 player games (where there is typically one dominant player), the weak player suffers a double curse: not only is the equilibrium policy far away from her ideal, but the costs she incurs to keep it there are increasing in her weakness. Indeed, in this case, equilibrium requires that both players expend the same amount of force in opposite directions. For the weaker player then, not only is the status quo further away but she also has to pay more to maintain it than does the dominant player.

We can interpret Equation 12 as stating that players have a downward sloping demand curve for violence. The interpretation is confirmed by Equation 13 which states that the cheaper violence is, the more that is employed in terms of magnitude.[6]

I now turn to consider how the amount of effort applied in equilibrium is affected by the *preferences* of a player. The next proposition states that *ceteris paribus,* players that are "extremist" in preference space are also "extremist" in effort. Players far from the center of a group expend more effort to try to change the equilibrium than centrist players.

Proposition 4 *In equilibrium, players far from the center of the group pay more than players closer to the center of the group, ceteris paribus.*

Proof. May be read for any two players directly from Equation 10. Furthermore, Differentiating Equation 10 with respect to x^i gives, on each dimension: $\partial \left[|x^i - \frac{\sum\limits_{i \in \mathcal{M}} \pi^i x^i}{\sum\limits_{i \in \mathcal{M}} \pi^i}|^2 \pi^i \right] / \partial x^i = \left[2\pi^i \frac{\sum\limits_{i \in \mathcal{M}} \pi^i - \pi^i}{\sum\limits_{i \in \mathcal{M}} \pi^i} (x^i_j - y^*_j) \right]_{j=1,2,\ldots n}$. Hence if on any dimension $x^i_j > (<) y^*_j$ then increasing x^i_j increases (decreases) the costs paid in equilibrium. ■

Finally, for comparison with more cooperative situations, it is useful to note that the sum of the utilities of all players in equilibrium under atomistic play in equilibrium in the stage game is given by:

[6] Strictly speaking, these statements refer to the "power" interpretation of π rather than the price interpretation. To get expressions for the price interpretation we need to differentiate with respect to $(1/\pi)$, or, equivalently, to multiply these expressions by $-(1/\pi)^2$.

$$\sum_{i\in\mathcal{M}} V^i = -\sum_{i\in\mathcal{M}} \left[x^i_j - \frac{\sum\limits_{i\in\mathcal{M}} \pi^i x^i}{\sum\limits_{i\in\mathcal{M}} \pi^i} \right]^2 - \sum_{i\in\mathcal{M}} \pi^i \rho(\mathcal{M}|\boldsymbol{\pi}) \tag{15}$$

If all players have equal weights, this reduces to $\sum\limits_{i\in\mathcal{M}} V^i = -(1 + \pi)|\mathcal{M}|\rho(\mathcal{M})$.

3.2 The Repeated Game with Myopic Players

So far we have considered the actions taken by agents that attempt to alter policy in a single time period. From Equation 5 however we can see that the process we are considering is described by a linear difference equation. Given some initial policy, y_0, each dimension has a solution in each period, s, given by:

$$y_{s,j} = \frac{\sum\limits_{k\in\mathcal{M}} \pi^k x^k_j}{\sum\limits_{k\in\mathcal{M}} \pi^k} + \left[\frac{1}{1 + \sum\limits_{k\in\mathcal{M}} \pi^k} \right]^s \left[y_{0,j} - \frac{\sum\limits_{k\in\mathcal{M}} \pi^k x^k_j}{\sum\limits_{k\in\mathcal{M}} \pi^k} \right] \tag{16}$$

This difference equation tells us all we need to know about the system, it allows us to predict policy at any point in time given some initial starting point. It also tells us that over time policy converges in a straight line to the steady state, given by $y_\infty = y^* = \frac{\sum_{k\in\mathcal{M}} \pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k}$. Furthermore, the rate of convergence is proportional to the distance of the policy from the steady state, where the factor of proportionality depends on the total absolute costs to all players of forcing a change. Summing over v^{i*}_t, we find:

$$\sum_{i\in\mathcal{M}} v^{i*}_t = (y^* - y_{t-1}) \frac{\sum\limits_{i\in\mathcal{M}} \pi^i}{\sum\limits_{i\in\mathcal{M}} \pi^i + 1} \tag{17}$$

If for example $\sum\limits_{k\in\mathcal{M}} \pi^k = 1$, then the process makes it half the way to equilibrium in every move. Note in particular that for a given steady state, the rate of convergence is independent of the ideal points of the players.

3.3 The Repeated Game with Sophisticated Players

In the last section I assumed that actors take actions to maximize their instantaneous gains, without regard for how their actions will affect future rounds of play. In this section I consider the possibility that 1) players do care about the future and 2) they are sophisticated in the sense that they take into account the effects of their actions today on their own actions and other players actions in future rounds of play.

Considering first the case where all players have the same strength, we find that under sophisticated play in a game of arbitrary length, the equilibrium is still given by the simple average of the ideals of the players and the path to equilibrium is still linear. However, the rate at which policy converges to equilibrium changes. I formalize these claims in the following proposition. The lemmas supporting the proof of this proposition are straightforward but tedious to prove. Lemma 1 is stated below but its proof and the supporting Lemma is left to the Appendix.

Lemma 1. *If a game with quadratic subutility functions and costs is iterated T times, where T is arbitrarily large, and all players are of equal strength, then the sum of all players' optimal actions, $\sum\limits_{i\in\mathcal{M}} v_t^{i*}$, in each period of play, $t = 1, 2, ..., T$, is given by a linear equation of the form: $\sum\limits_{i\in\mathcal{M}} v_t^{i*} = (\bar{x} - y_{t-1})k_t$, where $0 < k_t < 1$.*

Proof. See Appendix. ■

Proposition 5 *The equilibrium point in a game iterated T times, T arbitrarily large, where all players are sophisticated, are of equal strength, and have quadratic subutility functions and costs, is equivalent to the myopic equilibrium point: $y^* = \bar{x}$. Furthermore, policy converges linearly to $\bar{x}$ over time.*

Proof. *From Lemma 1 we have that in all periods $\sum\limits_{i\in\mathcal{M}} v_t^{i*} = (\bar{x} - y_{t-1})k_t$. But then if in any period t, $y_t = \bar{x}$ then y_{t+1} will be given by $y_{t+1} = y_t + \sum\limits_{i\in\mathcal{M}} v_{t+1}^{i*} = y_t + (y_t - y_t)k_{t+1} = y_t = \bar{x}$. And similarly for any y_s with $t < s$ □ T. This establishes the first part of the proposition. If in some time t, $y_t \neq \bar{x}$ then we have $y_{t+1} = y_t + (\bar{x} - y_t)k_{t+1}$ with (again from Lemma 1) $0 < k_t < 1$. But this implies that y_{t+1} is a convex combination of $\bar{x}$ and y_t and so y_t,y_{t+1} and $\bar{x}$ are collinear with y_{t+1} between y_t and $\bar{x}$. Similarly, all y_s with $t > s \geq T$ are collinear with y_s between y_{s-1} and $\bar{x}$.* ■

We have seen then that introducing sophisticated players does not greatly alter play in situations where all players are of equal strength. Play is however greatly affected when players are sophisticated and are not all of equal strength. In particular the myopic equilibrium point (the weighted average of the players' ideals) is not an equilibrium point in the game, and, moreover, there generally is no equilibrium point in this game.

Proposition 6 *A game iterated T times, T arbitrarily large, where all players are sophisticated, have heterogenous, generally distributed ideals, are not all of equal strength, and have quadratic subutility functions and costs, has no equilibrium point in any but the last period*

Proof. It is sufficient to show that with sophisticated players there is no point y_{T-2} such that with players playing equilibrium strategies $y_{T-2} = y_{T-1} = y_T$. Since we know from our study of the stage game that $y_{T-1} = y_T$ only if $y_{T-1} = y^* \equiv \frac{\sum_{k\in\mathcal{M}} \pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k}$, the only way that we could have $y_{T-2} = y_{T-1} = y_T$ is that, given $y_{T-2} = \frac{\sum_{k\in\mathcal{M}} \pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k}$, $\sum_{k=1}^{m} v^i_{j,T-1} = 0$ Hence we now consider $\sum_{k=1}^{m} v^i_{j,T-1}$. First define $\Pi = \sum_{k\in\mathcal{M}} \pi^k$ and $\breve{\Pi} = \sum_{k\in\mathcal{M}} \pi^k \pi^k$. In time $T-1$, forward-looking individuals choose a vector v^i_{T-1} to maximize $V^i_{T-1} = -(|x^i - y_{T-2} - \sum_{k=1}^{m} v^k_{T-1}|^2) - \frac{|v^i_{T-1}|^2}{\pi^i} - \delta(|x^i - y_T|^2) - \delta\frac{|v^i_T|^2}{\pi^i}$. Using backwards induction and Equation 5 they substitute $\frac{y_{T-1}+\Pi y^*}{1+\Pi}$ for y_T , and $\left[x^i - \frac{y_{T-1}+\Pi y^*}{1+\Pi}\right]\pi^i$ for v^i_T. Finally, players substitute $y_{T-2} + \sum_{k=1}^{m} v^k_{T-1}$ for y_{T-1}. The first order conditions for this problem yield (for each dimension, j):

$$v^i_{j,T-1} = \pi^i(x^i_j - y_{j,T-2} - \sum_{k\in\mathcal{M}} v^k_{j,T-1}) + \frac{\delta\pi^i(1+\pi^i)}{1+\Pi}(x^i_j - \frac{y_{T-2} + \sum_{k=1}^{m} v^k_{T-1} - \sum_{k\in\mathcal{M}} \pi^k x^k}{1+\Pi})$$

Summing this condition over the players and solving for $\sum_{k\in\mathcal{M}} v^k_{j,T-1}$ we find:

$$\sum_{k=1}^{m} v^i_{j,T-1} = \frac{\Pi(1+\Pi)^2+\delta\Pi(1-\breve{\Pi})}{(1+\Pi)^3+\delta(\Pi+\breve{\Pi})} y^* + \frac{\delta(1+\Pi)\breve{\Pi}}{(1+\Pi)^3+\delta(\Pi+\breve{\Pi})} \frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k} - \frac{\Pi(1+\Pi)^2+\delta(\Pi+\breve{\Pi})}{(1+\Pi)^3+\delta(\Pi+\breve{\Pi})} y_{T-2}$$

Now assuming $y_{T-2} = \frac{\sum_{k\in\mathcal{M}} \pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k}$ we have $\sum_{k=1}^{m} v^i_{j,T-1} = \frac{\delta\breve{\Pi}(1+\Pi)}{(1+\Pi)^3+\delta(\Pi+\breve{\Pi})}(\frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k} - y^*)$ and hence $\sum_{k=1}^{m} v^i_{j,T-2} = 0$ if and only if $\frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k} = \frac{\sum_{k\in\mathcal{M}} \pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k}$. This condition will be satisfied for generally distributed ideals if and only if $\pi^k = \pi$ for all $k \in \mathcal{M}$. [Note that this condition may be satisfied if $x^k = x$ for all $k \in \mathcal{M}$ and for some pathological joint distributions of π^k and x^k where $x^1 = \frac{x^2(\pi^2 \sum_{k\in\mathcal{M}} (\pi^k)^2 - (\pi^2)^2 \sum_{k\in\mathcal{M}} \pi^k) + \ldots + x^{|\mathcal{M}|}(\pi^{|\mathcal{M}|} \sum_{k\in\mathcal{M}} (\pi^k)^2 - (\pi^{|\mathcal{M}|})^2 \sum_{k\in\mathcal{M}} \pi^k)}{(\pi^1)^2 \sum_{k\in\mathcal{M}} \pi^k - \pi^1 \sum_{k\in\mathcal{M}} (\pi^k)^2}$, which will not hold for generally distributed ideals)] ■

Generally then with sophisticated players with asymmetric strengths we will not observe an equilibrium point. What then may we expect to happen in games with asymmetric players? As can be seen from the proof of the preceding Proposition, we may expect policy to tend towards the ideals of the strongest players. From the proof we may see for example that $\sum_{k=1}^{m} v^i_{j,T-1} = 0$ if and only if $y_{T-2} = \frac{\Pi(1+\Pi)^2 + \delta\Pi(1-\breve{\Pi})}{\Pi(1+\Pi)^2+\delta(\Pi+\breve{\Pi})} y^* + \frac{\delta(1+\Pi)\breve{\Pi}}{\Pi(1+\Pi)^2+\delta(\Pi+\breve{\Pi})} \frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k}$, that is if y_{T-2} lies on the line segment between y^* and a second weighted average of x^k, $k \in \mathcal{M}$, one that places greater weight on the ideals of stronger players (players that have a weight $\pi^i > \sum_{k\in\mathcal{M}} \pi^k\pi^k / \sum_{k\in\mathcal{M}} \pi^k$). If indeed $y_{T-2} = y^*$, then the stronger players pull the policy towards $\frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k}$ in time $T-1$, only for it to return again towards y^* in time T. In a similar manner we can show that in time $T-2$ strong players will act to pull policy towards a point given by $\frac{\sum_{k\in\mathcal{M}} \pi^k\pi^k\pi^k x^k}{\sum_{k\in\mathcal{M}} \pi^k\pi^k\pi^k}$, that is, a point that weights stronger players more heavily still.[7]

3.4 Generalization of the Atomistic Game to a Broader Class of Utility Functions

The results that have been produced so far have been derived by assuming a restrictive class of subutility functions: ones that are quadratic in the distance of policy from the ideal point. It is reasonable to ask how robust are these results to the particular functional form that has been chosen. In this section I provide evidence that the choice of the subutility function is not critical for these results. Here I only consider the location of the equilibrium point and I continue to maintain the assumption of quadratic costs; however I show that the central result that

[7] While I do not provide a solution for infinite horizon games, on the basis of the above discussion I conjecture that an equilibrium point will exists in these games and that it will lie closer to the ideal points of strong players than the weighted average.

the inefficient policy choice is given by the weighted utilitarian optimum turns out to be true for a very wide class of subutility functions.

Proposition 7 *If the subutility utility functions over a compact subset of* $\mathbb{R}^n$ *are smooth and concave and costs are quadratic in magnitude, then every equilibrium point is a weighted utilitarian welfare maximizing policy, where the weights are given by the relative strengths of the players.*

Proof. Assume that the status quo is located at some equilibrium point y. The conditions on $u(.)$ imply that for each player, i, the optimum vector of force applied by i, v^{i*}, given y and the forces applied by all other players is implicitly defined by the first order condition for utility maximization: $\frac{\partial u^i(y+v^1+v^2...+v^{i*}+...+v^m)}{\partial v^i} = \frac{1}{\pi^i}v^{i*}$. Summing such first order conditions over all $|\mathcal{M}|$ players, we have that in equilibrium $\sum\limits_{i\in\mathcal{M}} \pi^i \partial u^i(y + \sum\limits_{i\in\mathcal{M}} v^{i*})/\partial v^i - 2\sum\limits_{i\in\mathcal{M}} v^{i*} = 0$. By definition, however, if y is an equilibrium point, then, at y, $\sum\limits_{i\in\mathcal{M}} v^{i*} = 0$. Noting further that $\frac{\partial u^i(y+v^{1*}+v^{2*}...+v^i+...+v^{m*})}{\partial v^i} = \frac{\partial u^i(y+v^{1*}+v^{2*}...+v^{i*}+...+v^{m*})}{\partial y}$, we have that if y is an equilibrium point, then $\sum\limits_{i\in\mathcal{M}} \pi^i \frac{\partial u^i(y)}{\partial y} = 0$. But this is precisely the first order condition for maximizing the weighted utilitarian objective function $\sum\limits_{i\in\mathcal{M}} \pi^i u^i(y)$. ■

We have established then that for a very broad class of utility functions, a point is stable in the stage game only if it is the utilitarian welfare maximizing point. I now show for a more constrained class of utility functions that if, in the stage game, a point is a utilitarian welfare maximizing point, that it is a stable point.

Proposition 8 *If the subutility utility functions over a compact subset of* $\mathbb{R}^n$ *are smooth and concave and have the property that* $\frac{\partial^2 u^i(y)}{\partial y \partial y^T}$ *is negative definite for all* i*, and if costs are quadratic in magnitude, then a point* y *is an equilibrium point if and only if it is a utilitarian optimum.*

Proof. The "only if" part has already been established and so we turn to the "if" part. For notational convenience, let us define $\Delta_{n\times 1} \equiv \sum\limits_{i\in\mathcal{M}} v^{i*}(y)$. Also note that $\frac{\partial u^i(y+\Delta)}{\partial v^i} = \frac{\partial u^i(y+\Delta)}{\partial \Delta}$. We are given from the first order conditions of utility maximization, that $\Delta = \sum\limits_{i\in\mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta}$ and from the condition that y is a utilitarian maximum that $\sum\limits_{i\in\mathcal{M}} \pi^i \frac{\partial u^i(y)}{\partial \Delta} =$

0. Note first that since $\sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y)}{\partial \Delta} = 0$, $\Delta = 0$ satisfies $\Delta = \sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta}$. If $\Delta = 0$ is the *only* value of Δ that satisfies $\Delta = \sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta}$ then we have that $\Delta = 0$, necessarily, and hence that y is stable and we are done. I now show that given our assumptions on utility there is a unique solution to $\Delta = \sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta}$. Define $f(\Delta|y) = \sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta} - \Delta$ and note that $f(\Delta|y)$ is smooth and that its Jacobian matrix, $\frac{\partial f(\Delta|y)}{\partial \Delta}$, is negative definite. The Gale-Nikaido univalence theorem then guarantees that $f(\Delta|y)$ is injective and hence that there is a unique solution to $f(\Delta|y) = 0$. Such a solution however is a unique fixed point for $\sum_{i \in \mathcal{M}} \pi^i \frac{\partial u^i(y+\Delta)}{\partial \Delta}$. ■

Example 9 ***Quadratic utility with salience weights and possible non-separability across dimensions.*** *Assume utility takes the form $u^i(y) = (x^i - y)^T A^i (x^i - y)$ where A^i is any player-specific symmetric negative definite, possibly non-diagonal, $n \times n$ weighting matrix. Again, define $\Delta_{n \times 1} \equiv \sum_{i \in \mathcal{M}} v^{i*}(y)$. Since $\frac{\partial [(x^i - y - \Delta)^T A^i (x^i - y - \Delta)]}{\partial v^i} = -2A^i(x^i - y - \Delta)$ we have from the conditions for individual maximization that $\Delta = -2 \sum_{i \in \mathcal{M}} \pi^i A^i (x^i - y - \Delta)$. Hence Δ is given uniquely by $\Delta = [(I_{n \times n} - 2 \sum_{i \in \mathcal{M}} \pi^i A^i)]^{-1} \sum_{i \in \mathcal{M}} \pi^i A^i (x^i - y)$.[8] It follows then that if y is a utilitarian optimum, then $\sum_{i \in \mathcal{M}} \pi^i A^i (x^i - y) = 0$ and hence, $\Delta = 0$. Similarly, if $\Delta = 0$ then $\sum_{i \in \mathcal{M}} \pi^i A^i (x^i - y) = 0$ and hence y is a utilitarian optimum.*

3.5 Summary of Results from the Atomistic Game and an Example of a Game with Two Players

I now sum up the key results from this section informally. I follow this with an example of a game with two players in two dimensions.

[8] To check that the matrix $(I - 2 \sum_{i \in \mathcal{M}} \pi^i A^i)$ is invertible note that it is a linear combination of $|\mathcal{M}| + 1$ poisitive definite matrices and hence is itself positive definite and, therefore, (since a necessary condition for positive definiteness for a matrix is that its determinant be positive) non-singular.

Summary of Results

Equilibrium. The equilibrium stable point is unique and is given by a weighted average of the ideal points of the players, where the weights are determined by the relative strengths of the players.

Path to Equilibrium. The path to equilibrium is linear and in the myopic game satisfies a first order linear difference equation.

Path (In)Dependence. The equilibrium outcome (unlike the path to equilibrium) is independent of the starting position.

(In)Efficiency. Unless all players share the same ideal point and so $\rho(\mathcal{M}|\boldsymbol{\pi}) = 0$, the equilibrium strategies will be "socially wasteful," in the sense that costly energy will be spent without any gains in welfare. The waste is directly proportional to the dispersion of the group and increasing in the average level of strength of members of the group.

Extremist Activism. Players whose ideals are "extreme" relative to the rest of their group will apply more force to attempt to change the equilibrium, in equilibrium, than more centrist players.

Dominant Players. Dominant players enjoy an equilibrium closer to their ideals than weaker players, ceteris paribus, and pay less than weaker players, ceteris paribus to maintain the equilibrium.

Truthfulness. In equilibrium, players' strategies will be "truthful" in the sense that the forces they apply point directly to their true ideals from the equilibrium point (or, if the equilibrium point is the ideal of some player, then that player will apply a vector of length zero in equilibrium).

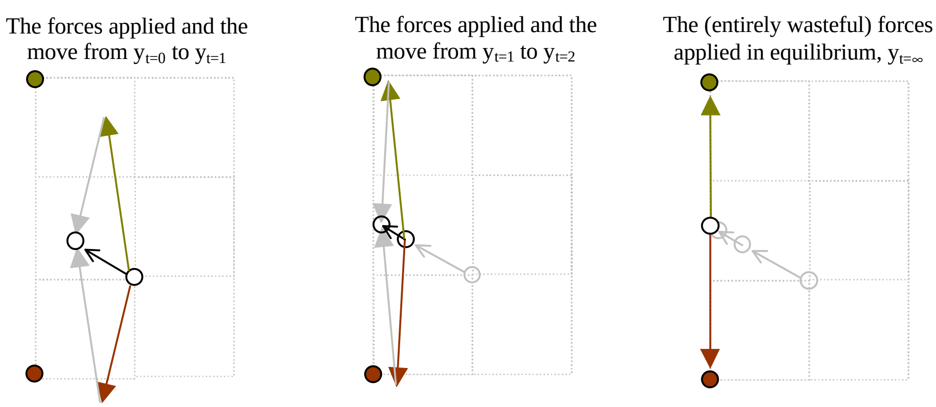
Sophisticated Players. If players are sophisticated then, in general, the equilibrium point of the repeated game is the same as that of the stage game (or the repeated game with myopic players) if and only if all players are of equal strength. If players are of divergent strength then there is in general no equilibrium point for a game of length $T > 2$.

Equivalence. The y^* that is picked out as a conflictual and costly equilibrium of the non-cooperative game is also the weighted utilitarian optimum where weights are given by the strength of players. It also lies inside the Pareto Set and so is in the core for consensus games. Furthermore this feature holds for all smooth concave subutility functions. Less restrictively, if either all players are of equal strength, or the "strengths" translate directly into votes, y^* also lies inside the core for a class of 64%-majority rule games.[9] Of course none of these cooperative solutions involve the inefficiencies needed here to sustain y^*.

[9] Andrew Caplin and Barry Nalebuff, "Aggregation and Social Choice: A Mean Voter Theorem" *Econometrica*, 59(1):1–23.1991. Although see also: Barry K. Ma , Jeffrey H. Weiss, "On the Invariance of a Mean Voter Theorem", *Journal of Economic Theory*, Vol. 66, 1995, pp. 264-274. pp. 264-274

Example

Assume that two players occupy positions $x^1 = (0,0)$ and $x^2 = (0,3)$, that the status quo is located at $y_{t=0} = (1,1)$ and that the players are symmetric in strength: $\pi^1 = \pi^2 = 1$. Under these assumptions $v^{1*}_{j,t=1} = \frac{[2x^1_j - y_j - x^2_j]}{3}$ and $v^{2*}_{j,t=1} = \frac{[2x^2_j - y_j - x^1_j]}{3}$. Hence we have $v^{1*}_{t=1} = \left(-\frac{1}{3}, -\frac{4}{3}\right)$ and $v^{2*}_{t=1} = \left(-\frac{1}{3}, +\frac{5}{3}\right)$. And the actual displacement is given by: $v^1_{t=1} + v^2_{t=1} = \left(-\frac{2}{3}, \frac{1}{3}\right)$. Hence the new status quo is: $y_{t=1} = y_{t=0} + v^1_{t=1} + v^2_{t=1} = \left(\frac{1}{3}, \frac{4}{3}\right)$. As we can see from the diagram below, this is an extremely inefficient way to get to $(\frac{1}{3}, \frac{4}{3})$ from $(1,1)$. We may repeat the process again at time $t = 2$ (and with $y_{t=1}$ as the status quo). The equilibrium force levels are given then by $v^{1*}_{t=2} = \left(-\frac{1}{9}, -\frac{13}{9}\right)$ and $v^{2*}_{t=2} = \left(-\frac{1}{9}, +\frac{14}{9}\right)$. The actual displacement is then given by $v^1_{t=2} + v^2_{t=2} = \left(-\frac{2}{9}, +\frac{1}{9}\right)$ and so the new status quo is $y_{t=2} = y_{t=1} + v^1_{t=2} + v^2_{t=2} = \left(\frac{1}{9}, \frac{13}{9}\right)$. In the limit $v^1_{t=\infty} + v^2_{t=\infty} = 0$ and $y_{t=\infty} = y^* = \left(0, \frac{3}{2}\right)$.



Note: In the figure, the status quo is given by a white dot. The ideals of two players, "red" and "green" are also marked with dots at (0,0) and (0,3). The forces applied at each stage in the game by the players are given by red and green arrows, and the resolution of the forces is marked.

Discussion. The example illustrates a number of interesting features about this process. Consistent with the general results from this section we find:

The limit of the process results in a stable policy (**Equilibrium**). At the equilibrium point, there are positive levels of expenditure. In the limit players keep exerting effort but their efforts to change policy offset each other exactly (**Inefficiency**). The path to equilibrium is linear and steps become shorter and shorter (**Path to Equilibrium**). Also, the limit point is a plausible bargaining outcome. It is the utilitarian optimum and the Nash bargaining solution to a game where the two players have the same attitudes to the status quo. Furthermore with transferable utility this is the general Coasian solution, since the Pareto optimal policy choice is unique (**Equivalence**).

Finally, the example is also instructive insofar as it illustrates how complex collective action problems can be easily accommodated by this framework. In this example players agree on the ideal policy along the first dimension but disagree about policy on the second dimension. They are both unhappy with policy along both dimensions. Their dilemma along the first dimension is a collective action problem: they agree on what actions should be taken but would each rather that the other player takes the action. In taking their actions they "underprovide" in the sense that they fail to take into account the positive externalities of their action for other players. Here the failure to engage in cooperative agreement means that effort is *underprovided* on the path to equilibrium. Along the second dimension however there is disagreement over ideals. Here the players "overprovide" insofar as they fail to internalize the negative externalities resulting from their actions on other players. In this case, particularly in equilibrium, effort is *overprovided* relative to the optimum. We see in the example that when the status quo is far away from the contract curve, a large share of the energy expended is actually used to move the policy towards the status quo; a small share is used trying to cancel out "robbing" effects by the other player. In contrast, when the status quo is very close to the contract curve, almost all of the energy is used to "cancel" the efforts made by the other player and almost none is used in making joint gains (there are few to be made!).

4 Exogenous Coalitional Structures

> I am a member of the Tutsi tribe by my mother, my father and my most remote ancestors [...] I am not sorry for having struggled alongside the Hutu. It is too late to be sorry. But if I had to do it again I would not do it. I would do everything except fight on behalf of a tribe other than mine, or to appear to be fighting on the side of any other tribe.
>
> - Boniface Kiraranganiya, *La vérité sur le Burundi* (in Lemarchand 1994 p58)

I now turn to the study of coalitions to see how the ability of subgroups to coordinate their actions affects outcomes. While in general we do not expect it to be an easy matter for any heterogenous group to produce a joint strategy I simplify matters in this section by assuming that players in a coalition, $\mathcal{C}^k$, can negotiate efficiently in that the strategies that they choose will be (Pareto) efficient in the sense that no other strategy would increase the welfare of any player(s) in the coalition without reducing the welfare of some other player(s) in the coalition. In many instances I also further simplify by assuming *Transferable Utility* (**TU**), that is, that players within a group can make side payments to each other and that these payments are valued equally by each member and increase the utility of any player linearly. At this stage I make no claims about the mechanisms used by groups to choose their outcome or regarding the relative bargaining strengths of players within a group. Hence I make no claims regarding how side payments are made among members of a group but only regarding the aggregate size of the pie for a group, a quantity determined in a TU game, purely by the choice of policy.

I begin by showing that coalitions can be represented by fictitious players with ideal points related to the ideal point of the individual members of the coalition and a strength related to the strength of the individual members of the coalition. This is established in the first subsection, doing so allows us quickly to extend results from our study of atomistic structures to structures with exogenous coalitions. I then turn to study the properties of equilibrium and the distribution of labor among members of a coalition. The grand coalitional structure is a special case of the structures considered here and so I use the results generated to compare positive and normative properties of the grand coalitional game with the atomistic game. Finally I relate coalitional structures to aggregate levels of violence, In doing so I link the theoretical work undertaken here with empirical work that has been done related the ethnic structuration of a country with aggregate levels of violence. The section ends with a summary of findings and an example illustrating some of the results from the model.

4.1 Little Leviathans: Representation of Coalitions with Fictitious Actors

Members of a coalition choose a set of actions $\{v^i\}_{i\in\mathcal{C}^k}$ that each member of the coalition takes in order to maximize some objective function of the coalition. In a game with transferable utility and no budget constraints, this objective function will be the summation of the utility of the members of the coalition. That is, the coalition will have a utilitarian objective function. In a game with non-transferable utility, the coalition may be concerned about the distributive effects of their actions and use some other objective function. Here I concentrate on the transferable utility case and derive other preliminary results only for the class of λ-weighted utilitarian objective functions.[10]

Assume for some arbitrary weighting vector λ, in time t, that each coalition $\mathcal{C}^k$ chooses a set of vectors $\{v_t^1, v_t^2....v_t^{|\mathcal{C}^k|}\}$ to maximize the $\lambda-$weighted sum of the utility of the members of the coalition:

$$\sum_{i\in\mathcal{C}^k}^{i}\lambda^i u_t^i = -\sum_{i\in\mathcal{C}^k}\lambda^i(|x^i - y_{t-1} - \sum_{h\in\mathcal{C}^k} v_t^h - \sum_{h\notin\mathcal{C}^k} v_t^h|^2) - \sum_{i\in\mathcal{C}^k}\frac{\lambda^i}{\pi^i}|v_t^i|^2$$

The first order condition for this problem for arbitrary dimension j and force enacted by player i in $\mathcal{C}^k$ is given by

$$v_{j,t}^i = \frac{\pi^i}{\lambda^i}\left[\sum_{h\in\mathcal{C}^k}\lambda^h(x_j^h - y_{j,t-1} - \sum_{h\in\mathcal{C}^k} v_{j,t}^h - \sum_{h\notin\mathcal{C}^k} v_{j,t}^h)\right] \tag{18}$$

Summing first order conditions from Equation 18 over the members of $\mathcal{C}^k$ and solving for $\sum_{i\in\mathcal{C}^k} v_{j,t}^i$ we may define:

$$v_t^{\mathcal{C}^k} \equiv \sum_{i\in\mathcal{C}^k} v_t^i = \frac{\sum_{i\in\mathcal{C}^k}\frac{\pi^i}{\lambda^i}}{1+\sum_{i\in\mathcal{C}^k}\frac{\pi^i}{\lambda^i}\sum_{i\in\mathcal{C}^k}\lambda^i}\sum_{i\in\mathcal{C}^k}\lambda^i(x^i - y_{t-1} - \sum_{i\notin\mathcal{C}^k} v_{t-1}^i) \tag{19}$$

[10] These functions may be used for studying games with non-transferable if we can assume that games without transferable utility may be reperesentable as games with transferable λ-weighted utility. See Myerson 1991.

"$v^{\mathcal{C}^k}$" then fully describes the joint actions taken by a group, as if it were a unitary actor. Indeed it turns out that we can fully represent the actions a coalition by the actions of a fictitious player, $\mathcal{C}^k$, with strength: $\pi^{\mathcal{C}^k} \equiv \sum_{i\in\mathcal{C}^k} \frac{\pi^i}{\lambda^i} \sum_{i\in\mathcal{C}^k} \lambda^i$ and ideal point $x^{\mathcal{C}^k} \equiv \frac{\sum_{k\in\mathcal{C}^k} \lambda^i x^i}{\sum_{i\in\mathcal{C}^k} \lambda^i}$. To check this, note that an individual player with this ideal and strength acting to satisfy an individual's first order condition (given in Equation 3) will choose

$$v_{j,T-1}^{\mathcal{C}^k} = \frac{\sum_{i\in\mathcal{C}^k} \frac{\pi^i}{\lambda^i} \sum_{i\in\mathcal{C}^k} \lambda^i}{1+\sum_{i\in\mathcal{C}^k} \frac{\pi^i}{\lambda^i} \sum_{i\in\mathcal{C}^k} \lambda^i} \frac{\sum_{i\in\mathcal{C}^k} \lambda^i x_j^i}{\sum_{i\in\mathcal{C}^k} \lambda^i} - \frac{\sum_{i\in\mathcal{C}^k} \frac{\pi^i}{\lambda^i} \sum_{i\in\mathcal{C}^k} \lambda^i}{1+\sum_{i\in\mathcal{C}^k} \frac{\pi^i}{\lambda^i} \sum_{i\in\mathcal{C}^k} \lambda^i} (y_{j,T-1} + \sum_{i\notin\mathcal{C}^k} v_{j,T-1}^i)$$

which is equivalent to the action taken by a coalition as reported in Equation 19.

Hence the coalition acts like an individual and so we may use the results from our study of atomistic structures in this section also. If for example, for all i, $\lambda^i = \pi^i$ then the coalition acts like a player with strength of $|\mathcal{C}^k| \sum_{i\in\mathcal{C}^k} \pi^i$ and ideal $\frac{\sum_{i\in\mathcal{C}^k} \pi^i x^i}{\sum_{i\in\mathcal{C}^k} \pi^i}$; if instead, for all i, λ^i=λ then the coalition acts like a player with strength of $|\mathcal{C}^k| \sum_{i\in\mathcal{C}^k} \pi^i$ and ideal $\bar{x}$. Furthermore the fictitious player, $\mathcal{C}^k$, not only acts like an individual, but he is "representative" of the members of the coalition from which he derives in the following ways. The costs he incurs are exactly the average of the costs of the individual members of the group; and his utility is the average utility of the members of the group, plus a bonus that is given by the degree of dispersion of the group. If all members of a coalition have the same ideals, then the utility of the fictitious player is exactly that of the individual members of the group. If the group however is dispersed, then the fictitious player does better than the individual members of the group. These claims are formalized in the next proposition.

Proposition 10 *The costs incurred by the fictitious player representing a coalition $\mathcal{C}^k$, $\frac{|v_{T-1}^{\mathcal{C}^k}|^2}{\pi^{\mathcal{C}^k}}$ equal the average costs incurred by the individual players in the coalition. That is $\frac{|v_{T-1}^{\mathcal{C}^k}|^2}{\pi^{\mathcal{C}^k}} = \sum_{i\in\mathcal{C}^k} \frac{|v_{T-1}^i|^2}{\pi^i}/|\mathcal{C}^k|$. Furthermore, the policy utility of the fictitious player $-|(x^{\mathcal{C}^k} - y)|^2$ is given by the* *average policy utility of the members of the coalition,* $\sum_{i\in\mathcal{C}^k}\frac{|x^i-y|^2}{|\mathcal{C}^k|}$ *plus the dispersion of the players; that is:* $-|x^{\mathcal{C}^k}-y|^2=-\sum_{i\in\mathcal{C}^k}\frac{|x^i-y|^2}{|\mathcal{C}^k|}+\rho(\mathcal{C}^k)$*.*

Proof. Since from Equation 10, each $v^i_{j,T-1}$, $i\in\mathcal{C}^k$ may be written $v^i_{j,T-1}=\pi^i\theta_j$ for some constant θ_j, it follows that $\sum_{i\in\mathcal{C}^k}v^i_{j,T-1}=\theta_j\sum_{i\in\mathcal{C}^k}\pi^i$ and so $v^i_{j,T-1}=\pi^i\frac{\sum_{i\in\mathcal{C}^k}v^i_{jT-1}}{\sum_{i\in\mathcal{C}^k}\pi^i}$ and $\frac{|v^i_{T-1}|^2}{\pi^i}=\frac{\pi^i|\sum_{i\in\mathcal{C}^k}v^i_{T-1}|^2}{\left[\sum_{i\in\mathcal{C}^k}\pi^i\right]^2}$. It follows immediately that $\sum_{i\in\mathcal{C}^k}\frac{|v^i_{T-1}|^2}{\pi^i}=|\sum_{i\in\mathcal{C}^k}v^i_{T-1}|^2\frac{\sum_i\pi^i}{\left[\sum_{i\in\mathcal{C}^k}\pi^i\right]^2}=|\sum_{i\in\mathcal{C}^k}v^i_{T-1}|^2\frac{1}{\sum_{i\in\mathcal{C}^k}\pi^i}=$ $|\mathcal{C}^k|\frac{|v^{\mathcal{C}^k}_{T-1}|^2}{\pi^{\mathcal{C}^k}}$.

For the second part of the proposition we write

$(x^{\mathcal{C}^k}_j-y_j)^2=\sum_{i\in\mathcal{C}^k}(y_j^2-2y_jx^{\mathcal{C}^k}_j+(x^{\mathcal{C}^k}_j)^2)/|\mathcal{C}^k|=\sum_{i\in\mathcal{C}^k}(y_j^2-2y_jx^{\mathcal{C}^k}_j-(x^{\mathcal{C}^k}_j)^2+2x^{\mathcal{C}^k2}_j)/|\mathcal{C}^k|$

making use of the fact that $\sum_{i\in\mathcal{C}^k}\theta x^i_j/|\mathcal{C}^k|=\theta x^{\mathcal{C}^k}_j=\sum_{i\in\mathcal{C}}\theta x^{\mathcal{C}^k}_j/|\mathcal{C}^k|$ we then have that

$$\begin{aligned}(x^{\mathcal{C}^k}_j-y_j)^2&=\sum_{i\in\mathcal{C}^k}(y_j^2-2y_jx^i_j-(x^{\mathcal{C}^k}_j)^2+2x^{\mathcal{C}^k}_jx^i_j)/|C|\\&=\sum_{i\in\mathcal{C}^k}(x^i_j-y_j)^2/|\mathcal{C}^k|-\sum_{i\in\mathcal{C}^k}(x^i_j-x^{\mathcal{C}^k}_j)^2/|\mathcal{C}^k|\end{aligned}$$

Hence $\sum_j(x^{\mathcal{C}^k}_j-y_j)^2=\sum_{i\in\mathcal{C}^k}\sum_j(x^{\mathcal{C}^k}_j-y_j)^2/|\mathcal{C}^k|-\sum_j\sum_{i\in\mathcal{C}^k}(x^i_j-x^{\mathcal{C}^k}_j)^2/|\mathcal{C}^k|$ and hence $-|x^{\mathcal{C}^k}-y|^2=-\sum_{i\in\mathcal{C}^k}|x^i-y|^2/|\mathcal{C}^k|+\rho(\mathcal{C}^k)$. ■

This last proposition will allow us to extend our results from atomistic structures to more complex structures. For normative comparisons it will also be useful to be able to write an expression for the *value* of a coalition. This is given by the joint utility of the set of players in a coalition: $V^{\mathcal{C}^k}=-\sum_{i\in C}|x^i-y|^2-\sum_{i\in C}\frac{|v^i_{T-1}|^2}{\pi^i}$. Using the values derived above, this can be represented simply as:

$$V^{\mathcal{C}^k}=|\mathcal{C}^k|\left[-|x^{\mathcal{C}^k}-y|^2-\frac{|v^{\mathcal{C}^k}_{T-1}|^2}{\pi^f}-\rho(\mathcal{C}^k)\right]\tag{20}$$

4.2 Characterizing Equilibrium with Coalitions

We now use the fact that we can represent coalitions with single fictitious players in order to characterize equilibrium with coalitions. To do so we shall discuss the degree of dispersion of a coalitional structure $\rho(\mathcal{C})$, which we interpret as the dispersion of the ideals of the fictitious players $\{x^{\mathcal{C}^k}\}_{k=1}^{|\mathcal{C}|}$ in $\mathcal{C}$.

The next proposition describes the equilibrium point and the total quantity of wasted effort in equilibrium under exogenous coalitional structures.

Proposition 11 *If the population is partitioned into g rival coalitions, $\mathcal{C}^1, \mathcal{C}^2 ... \mathcal{C}^g$ and utility is transferable, then the stable point to the simple game is given by* $y^* = \frac{\sum_{k=1}^{g}\left[\sum_{i\in\mathcal{C}^k}\pi^i \sum_{i\in\mathcal{C}^k} x^i\right]}{\sum_{k=1}^{g}|\mathcal{C}^k|\sum_{i\in\mathcal{C}^k}\pi^i}$. *The average cost born by players is given by* $\sum_{k=1}^{g}\pi^{\mathcal{C}^k}\frac{\rho(\mathcal{C}|(\pi^{\mathcal{C}^{\mathbf{k}}})_{\mathcal{C}^k\in\mathcal{C}})}{|\mathcal{M}|}$.

Proof. We prove the proposition using the ideals and powers for the fictitious players corresponding to each coalition. We know from Equation 7 that in a game with g players, the stable policy is given by $y^* = \frac{\sum_{k=1}^{g}\pi^k x_j^k}{\sum_{k=1}^{g}\pi^k}$. In the simple TU game, the power of the fictitious player is given by $\pi^k = |\mathcal{C}^k| \sum_{i\in\mathcal{C}^k}\pi^i$ and his ideal is given by $x^k = \frac{\sum_{i\in\mathcal{C}^k} x^i}{|\mathcal{C}^k|}$. Substituting these quantities into Equation 7 yields $y^* = \frac{\sum_{k=1}^{g}\pi^{\mathcal{C}^k}x^{\mathcal{C}^k}}{\sum_{k=1}^{g}\pi^{\mathcal{C}^k}} = \frac{\sum_{k=1}^{g}\left[\sum_{i\in\mathcal{C}^k}\pi^i \sum_{i\in\mathcal{C}^k} x^i\right]}{\sum_{k=1}^{g}|\mathcal{C}^k|\sum_{i\in\mathcal{C}^k}\pi^i}$. Now letting $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2 ... \mathcal{C}^g\}$ denote the set of coalitions and making use of Equation 11, we know that the average costs born by the fictitious players is given by $\sum_{k=1}^{g}\pi^{\mathcal{C}^k}\frac{\rho(\mathcal{C}|(\pi^{\mathcal{C}^{\mathbf{k}}})_{\mathcal{C}^k\in\mathcal{C}})}{|\mathcal{C}|}$.Hence the average costs born to players is given by $\sum_{k=1}^{g}\pi^{\mathcal{C}^k}\frac{\rho(\mathcal{C}|(\pi^{\mathcal{C}^{\mathbf{k}}})_{\mathcal{C}^k\in\mathcal{C}})}{|\mathcal{M}|}$. ■

It is worth noting that the stable point is still a weighted average of the ideals of the players, as it was in the atomistic game. However in the coalitional game the weight attached to an individual, h's ideal, $h \in \mathcal{C}^k$

is given by $\frac{\sum\limits_{i\in\mathcal{C}^k}\pi^i}{\sum\limits_{k=1}^{g}|\mathcal{C}^k|\sum\limits_{i\in\mathcal{C}^k}\pi^i}$ rather than $\frac{\pi^h}{\sum\limits_{i\in\mathcal{M}}\pi^i}$; a quantity that depends on the number of players in player h's coalition and the strengths of each of them.

Using the same logic we can also infer that the path to equilibrium will be linear for myopic players and that a game of length T will have a stable point in periods other than the last if and only if $\pi^k = |\mathcal{C}^k|\sum\limits_{i\in\mathcal{C}^k}\pi^i = |\mathcal{C}^h|\sum\limits_{i\in\mathcal{C}^h}\pi^i$ for all $\mathcal{C}^k$, $\mathcal{C}^h$ in $\mathcal{C}$.

4.3 Distributive Aspects Within Groups

In this section I have been assuming that members of a coalition bargain efficiently. However I am silent with regards to the manner in which they distribute benefits among their members. Under the assumption of transferable utility, the distribution of benefits will not affect the actions taken by players, only the pattern of monetary transfers between members of a coalition. However, without making further assumptions regarding the mechanisms used by groups, or the transfers made between groups, we can make statements regarding the division of labor within groups. These are summarized for TU and non-transferable utility games that may be represented as λ-weighted TU games in the following proposition.

Proposition 12 ***The Assignment of Labor Within Groups.*** *When coalitions form, all members apply force in the same direction, and the load assigned to each is positively related to their strength and negatively related to their λ-weighting in the coalitional decision rule. Furthermore if a weighted utilitarian rule is used, with weights given by strengths of players, then all players pull equally.*

Proof. If for some arbitrary weighting vector λ, in time t, each coalition $\mathcal{C}^k$ chooses a set of vectors $\{v_t^1, v_t^2....v_t^{|\mathcal{C}^k|}\}$ to maximize $\sum\limits_{i\in\mathcal{C}^k}^{i}\lambda^i u_t^i = -\sum\limits_{i\in\mathcal{C}^k}\lambda^i(|x^i - y_{t-1} - \sum\limits_{h\in\mathcal{C}^k}v_{t-1}^h - \sum\limits_{h\notin\mathcal{C}^k}v_{t-1}^h|^2) - \sum\limits_{i\in\mathcal{C}^k}\frac{\lambda^i}{\pi^i}|v_{t-1}^i|^2$ then the first order condition for this problem for arbitrary dimension j and force enacted by player i in $\mathcal{C}^k$ is given by

$$v_{t-1}^i = \frac{\pi^i}{\lambda^i}\left[\sum_{h\in\mathcal{C}^k}\lambda^h(x^h - y_{t-1} - \sum_{h\in\mathcal{C}^k}v_{t-1}^h - \sum_{h\notin\mathcal{C}^k}v_{t-1}^h)\right] \tag{21}$$

The vector term in square brackets in Equation 21 is constant for all members of $\mathcal{C}^k$. This is establishes the first part of the proposition; the rest of the proposition follows from the fact that the scalar multiplier in Equation 21 is given by $\frac{\pi^i}{\lambda^i}$. ■

We see then that all players in a coalition pull in a single direction; the size of the pull is positively related to the strength of the players and negatively related to their weighting in the coalitional decision rule.

4.4 The Grand Coalition

The grand coalition is the special case of a coalitional structure where all players may be bound by a consistent set of contracts. I now compare outcomes under a grand coalitional structure and outcomes under the atomistic structure, first in terms of the policy chosen and second in terms of the aggregate gain in utility induced by the introduction of institutions to enforce contracts.

If $\mathcal{C}^k$ is the *grand coalition*, then, using Equation 19, $\sum\limits_{i\in\mathcal{M}} v_{j,t}^k = \frac{\sum\limits_{i\in\mathcal{M}}\frac{\pi^i}{\lambda^i}\sum\limits_{k\in\mathcal{M}}\lambda^k(x^k-y_{t-1})}{1+\sum\limits_{i\in\mathcal{M}}\frac{\pi^i}{\lambda^i}\sum\limits_{k\in\mathcal{M}}\lambda^k} = 0$ if and only if $\sum\limits_{k\in\mathcal{M}}\lambda^k(x^k - y_{t-1}) = 0$. That is, a point is a stable point if and only if it is a weighted average of the ideals, where the weight is given by the vector λ. With transferable utility, the stable point is the simple average. Furthermore, it is easy to confirm that if y_{t-1} is a stable point and there is a grand coalition, then, $v_{j,t-1}^i = 0$ for all i. We note:

1. That the point chosen by the grand coalition will be the same as the stable point in the non-cooperative game, if and only if the decision rule used by the committee is weighted utilitarian, with weights given by strength (a special case of which is the simple utilitarian rule in situations where all players are of equal strength).

2. That the powers, π^i have no affect on the location of the stable point in the TU game and affect the location of the stable point in non-transferable utility games only indirectly via the choice of λ.

3. That if all players have the same powers, then the policy outcome will be the same in a TU game with a grand coalitional structure as under the atomistic structure.

I now turn to consider efficiency losses under the atomistic structure. Using Equation 20 we have that the value of the grand coalition is

$$V^{\{\mathcal{M}\}} = -|\mathcal{M}|\rho(\mathcal{M}) \tag{22}$$

It is useful to compare this to the sum of the values of the individual players under the atomic structure. Using Equation 15 we find that the gain in the sum of utilities is given by:

$$V^{\{\mathcal{M}\}} - \sum_{i\in\mathcal{M}} V^i = \left[\sum_{i\in\mathcal{M}} \left[x_j^i - \frac{\sum\limits_{i\in\mathcal{M}} \pi^i x^i}{\sum\limits_{i\in\mathcal{M}} \pi^i} \right]^2 - |\mathcal{M}|\rho(\mathcal{M}) \right] + \sum_{i\in\mathcal{M}} \pi^i \rho(\mathcal{M}|\boldsymbol{\pi}) \tag{23}$$

If the π^i are not uniform then this represents a double gain in the simple sum of the utility of the group: first a gain from having an outcome that is at the utilitarian optimum rather than at the weighted utilitarian optimum. This gain is represented by the first term in parentheses in Equation 23. It represents the utility gain from *financial* compensation to strong players rather than compensation through *policy distortion* and is strictly positive unless all players have equal weights. The second gain follows from the saved expenditure in equilibrium, and is represented by the second term on the right hand side of Equation 23.

4.5 Effects of Coalition Structures

Coalitional structures affect total costs in two distinct ways. One relates to the distribution of interests within a coalition, the other to the distribution of interests across coalitions.

Coalitions allow players to internalize the effects of their actions on other players in the same coalition. In doing so, members in a coalition increase the effort put into actions that are mutually beneficial and they reduce the effort put into actions that impose negative externalities on other coalition members. Whether this increases the total effort undertaken by a coalition or not will depend on the homogeneity of the members of a coalition. If members of a coalition have common interests, they will increase the level of their activity, if they have strongly competing interests then they will reduce the level of their activity. So much for the distribution of interests within coalitions.

We have seen that in the game *between* coalitions, the aggregate level of effort applied in equilibrium depends positively on the dispersion of the ideals of the fictitious players representing the coalitions. Hence more polarized coalitional structures will be related to higher aggregate levels of violence.

In all cases however, the power of the coalition will, we have seen, be greater than the sum of the power of the individual members of the coalition. A less fragmented coalitional structure may increase the level of violence by increasing the average level of strength, but it may reduce it by decreasing the degree of dispersion across coalitions. These effects then work in different ways and are worth disentangling.

Comparing two coalitional structures, $\mathcal{C}$ and $\mathcal{D}$ we say that $\mathcal{C}$ is "**coarser**" than $\mathcal{D}$ and $\mathcal{D}$ is "**finer**" than $\mathcal{C}$ when the coalitions in $\mathcal{C}$ are unions of the coalitions in $\mathcal{D}$. And we say "strictly coarser" or "strictly finer" when $\mathcal{C} \neq \mathcal{D}$. Coarser partitionings are less fragmented in the sense that the number of bilateral bargains that can be made are larger under a coarse partitioning than under a finer one. $\{\mathcal{M}\}$ is the coarsest possible partitioning in our game and $\langle\mathcal{M}\rangle$ is the finest possible partitioning. We might expect then that coarser partitionings reduce the level of violence in a polity firstly because of the increased coverage of institutions and secondly since we may expect coarser partitions to reduce the level of dispersion between groups. However, the next proposition shows that if a partitioning of $\mathcal{C}$ that is coarser than some other partitioning $\mathcal{D}$, fails to reduce the dispersion of the fictitious players in a game, then the aggregate level of violence is strictly higher under the coarser coalitional structure than under the finer structure.

Proposition 13 *Coalitional structures that do not reduce agent dispersion increase conflict (as measured by the total cost of the conflict) relative to finer coalitional structures. More formally, if a coalitional structure* $\mathcal{C}$ *, is strictly coarser than* $\mathcal{D}$ *but* $\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^{\mathbf{k}}})_{\mathcal{C}^k\in\mathcal{C}}) = \rho(\mathcal{D}|(\boldsymbol{\pi}^{\mathcal{D}^{\mathbf{k}}})_{\mathcal{D}^k\in\mathcal{D}}) = \bar{\rho}$ *then* $\sum\limits_{i\in\mathcal{M}} \frac{(v^i|\mathcal{C})^2}{\pi^i} > \sum\limits_{i\in\mathcal{M}} \frac{(v^i|\mathcal{D})^2}{\pi^i}$.

Proof. The proposition follows from the fact with $\mathcal{C}$ strictly coarser than $\mathcal{D}$ $\sum\limits_{\mathcal{C}^k\in\mathcal{C}} \pi^{\mathcal{C}^k} = \sum\limits_{\mathcal{C}^k\in\mathcal{C}} \left|\bigcup\limits_{\mathcal{D}^k\in\mathcal{C}^k}\right| \sum\limits_{\mathcal{D}^k\in\mathcal{C}^k} \sum\limits_{i\in\mathcal{D}^k} \pi^i$ is strictly greater than $\sum\limits_{\mathcal{D}^k\in\mathcal{D}} \pi^{\mathcal{D}^k} = \sum\limits_{\mathcal{D}^k\in\mathcal{D}} |\mathcal{D}^k| \sum\limits_{i\in\mathcal{D}^k} \pi^i$. Hence, total costs under the coalitional structure $\mathcal{C}$, $\bar{\rho}\sum\limits_{k=1}^{g} \pi^{\mathcal{C}^k}$ exceed those under the atomic structure $\mathcal{D}$, $\bar{\rho}\sum\limits_{\mathcal{D}^k\in\mathcal{D}} \pi^{\mathcal{D}^k}$. ■

The next proposition shows conditions under which the policy outcome will be invariant to the coalitional structure. Even in these cases however, the aggregate level of expenditure is conditional upon the coalitional structure.

Proposition 14 *If $\mathcal{M}$ is partitioned into some set of coalitions, $\mathcal{C}$, such that $\sum_{i\in\mathcal{C}^k} \pi^i = \sum_{i\in\mathcal{C}^h} \pi^i$ for all $\mathcal{C}^h$, $\mathcal{C}^k$ in $\mathcal{C}$, (as for example if all coalitions contain the same number of players and all players have equal strength) then a) the stable point will be independent of the form of $\mathcal{C}$ but b) the costs born in equilibrium will be sensitive to the composition of the coalition.*

Proof. Since $y^* = \frac{\sum_{k=1}^{g}\left[\sum_{i\in\mathcal{C}^k}\pi^i \sum_{i\in\mathcal{C}^k} x^i\right]}{\sum_{k=1}^{g}|\mathcal{C}^k| \sum_{i\in\mathcal{C}^k}\pi^i}|C|$ and $\sum_{i\in\mathcal{C}^k} \pi^i = \sum_{i\in\mathcal{C}^h} \pi^i$ for all $\mathcal{C}^h$, $\mathcal{C}^k$ in $\mathcal{C}$, we have $y^* = \frac{\sum_{k=1}^{g}\left[\sum_{i\in\mathcal{C}^k} x^i\right]}{\sum_{k=1}^{g}|\mathcal{C}^k|} = \sum_{i\in\mathcal{M}}\frac{x^i}{|\mathcal{M}|} = \bar{x}$. This establishes the first part of the proposition. However, the total costs paid in equilibrium is given by $\sum_{k=1}^{g}\pi^{\mathcal{C}^k}\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^\mathbf{k}})_{\mathcal{C}^k\in\mathcal{C}})$. The latter part of this expression depends on the specific form of the partitioning, $\mathcal{C}$. ■

4.6 How the Theory Can Inform Empirical Work on Ethnicity and Violence

In this section I further investigate the effects of coalitional structures by calculating a measure of the aggregate level of conflictual actions taken in equilibrium for an exhaustive set of coalitional structures for a polity. I then relate the results from this investigation to empirical work that links ethnolinguistic fragmentation to aggregate levels of violence.

I consider a polity composed of ten identical agents, each with an ideal point, x^i, drawn from some common distribution with finite mean and variance and each with a common strength, π, normalized to unity.[11] For each of the 42 possible coalition structures containing 10 players I calculate the expected level of conflictual actions taken in equilibrium. This I treat as a proxy for the level of violence resulting from the failure of players to contract across coalitions. The expected level of violence in equilibrium in each coalitional structure is calculated as $\sum_{\mathcal{C}^k\in\mathcal{C}}\sum_{i\in\mathcal{C}^k}|\mathcal{C}^k|\pi^i \times \mathsf{E}\left[\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^\mathbf{k}})_{\mathcal{C}^k\in\mathcal{C}})\right]$ where $\mathsf{E}\left[\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^\mathbf{k}})_{\mathcal{C}^k\in\mathcal{C}})\right]$ denotes the

[11] Alternatively we could think of a population divided into deciles. Coalitions then are constrained to comprise some integer number of deciles of the population. Hence one coalition may contain 90% of the population while another contains the remaining 10%. The symmetry of the players and the restriction that coalitions contain population shares in increments of 10% restricts the number of strategically different coalitional structures to just 42. These 42 coalitional structures are listed in the Appendix.

expected weighted variance of the means of the ideal points of the coalitions. A table in the Appendix reports the level of aggregate costs of conflictual actions associated with each coalitional structure.

The findings from these calculations indicate that the most "violent" forms of interaction are likely to occur in polities where the population is divided into a small number of approximately equal sized groups. The least violent structures are those where either players are divided into many small groups or they are mostly contained in one large group. The reason is this. When players are divided into many small groups they fail to combine their strength in the manner required to produce high levels of violence. When players are largely contained in just one group, the group certainly will have much combined strength at its disposal but the (weighted) dispersion between coalitions will generally be small. The large group will be able to resolve much of the conflict in the polity internally and cooperatively. Situations where there are small numbers of large coalitions are the most explosive since here the strengths of the members will be combined and there is still the possibility for considerable dispersion between the ideals of the coalitions.

These results can constructively engage the empirical literature that relates ethnolinguistic fractionalization to violence. Much of this empirical literature (see for a recent treatment Elbadawi and Sambanis 2000) uses a summary statistic, "**ELF**" to capture the ethnolinguistic coalitional structure of a polity. ELF is defined as the probability that two members drawn randomly from a population will not come from the same ethnolinguistic group. It is therefore given (for large groups) by ELF$=1-\sum_{\mathcal{C}^k\in\mathcal{C}}\left[\frac{|\mathcal{C}^k|}{|\mathcal{M}|}\right]^2$, where $|\mathcal{C}^k|$ is the number of individuals in coalition (ethnolinguistic group) $\mathcal{C}^k$, given coalitional structure $\mathcal{C}$, and $|\mathcal{M}|$ denotes the total population of the polity.

Empirical work that studies the effects of ELF is rarely well informed by theory. It is not always clear first why we would expect the measure to have any political relevance and second, assuming ELF is relevant, what sort of relationship we should expect it to have with levels of violence.

With regard to the first question, one motivation for expecting that ELF will be related to violent action derives from the conjecture that the ethnolinguistic structures of a country describes to some extent the manner in which disputes may be settled within groups and between groups. If indeed organization is more easily achieved within ethnolinguistic groups than across them, then the ethnolinguistic structure of a polity corresponds well to the coalitional structures discussed in this paper.

With regard to the second problem, there has been found to be a quadratic relation in the data between ELF and measures of civil warfare.[12] The frequency of civil conflicts is found to be at its lowest in either very homogenous societies or in highly fractionalized societies. Societies where violence is most likely are those with scores in a middle range. Explanations for this quadratic relationship turn on the idea that middling values of ELF capture societies that are "polarized". These explanations seem to accord well with the theory-based relationships that we establish here between coalitional structures and aggregate levels of violence.

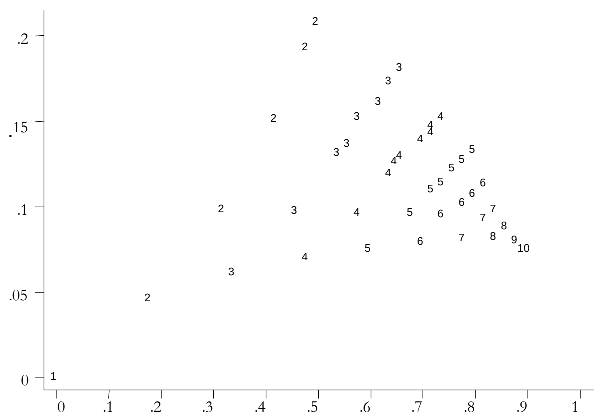
I argue however that one reason why the empirical results seem to accord well with the theoretical predictions of this model is that ELF itself is frequently misinterpreted. The claim that middling values of ELF correspond to societies in which there is "polarization" between groups, producing rivalry, represents a double misinterpretation. First it represents a misinterpretation of what ELF measures, since it ignores the fact that a single value for ELF may correspond to many substantively different coalitional structures (a point I discuss more below). Second it represents a misinterpretation of what polarization means insofar as it treats polarization as a function of relative group sizes rather than of the distribution of preferences across groups.[13]

In fact the theory developed here does not predict a quadratic relationship between ELF and aggregate levels of violence. To see this I calculated the value of ELF that corresponds to each of the 42 coalitional structures considered in this section. These values are reported in a table in the Appendix. The figure below plots predicted levels of violence against ELF for each of the coalitional structures considered (each number in the graph corresponds to a datapoint, the numbers denote the number of coalitions contained in the coalitional structure for the corresponding datapoint). It is worthwhile drawing attention to two features of the graph.

[12] Although more recent work by Sambanis (2001) suggests that this relationship is more likely to be linear for identity wars.

[13] The literature on polarization (see for example Deutsch (1971) or Esteban and Ray (1994)) treats polarization as a measure of the degree of heterogeneity between more or less homogenous groups where heterogeneity and homogeneity are defined over some sort of a preference or interest space. Esteban and Ray construct a measure of polarization for cases where players have preferences over a single dimension. This, and related work is primarily intended to capture degrees of agreement or disagreement across groups, not simply relative sizes of different groups. Despite Esteban and Ray's emphasis on the pattern of preferences, Collier and Hoe- er (2001) and Reynal-Querol (2000) both attempt to produce an Esteban and Ray type measure using only data on group size, claiming that their measure nonetheless measures polarization. In doing so however they effectively had to *assume* homogeneity of preference within groups and heterogeneity across groups.

First, if a regression is run on the levels of violence predicted by this model with ELF and ELF-squared on the right hand side, a concave quadratic relationship is found to be statistically significant, peaking at a similar point to that of the relationship found in empirical work. The theory then provides a theoretically based rationale for why we may expect to observe a quadratic relationship between ethnolinguistic structure and violence in the data.



This figure shows the mapping from ELF to the predicted sum of costs expended to force a change to an equilibrium point for a given coalitional structure. The points are labelled with an indicator of the number of coalitions in the corresponding coalitional structure.

Second, although we may find a quadratic relationship, we can see from the figure that this relationship is largely spurious. In the figure, the predicted level of violence is not a *function* of ELF: for any level of ELF there may be multiple levels of violence - depending on other aspects of the coalitional structure. In particular, middling values of ELF are not associated unambiguously with high levels of violence. The reason for the seeming ambiguity lies not with an indeterminate theory but with the measure ELF. ELF does not capture all politically relevant aspects of social fragmentation that are shown in this paper to be relevant for policy outcomes. Coalition structures that are very different from a political point of view may nonetheless have the same ELF value. As an example, a coalitional structure in which the population is divided into two equally sized groups receives approximately the same ELF score (.5) as a population in which one group contains 70% of the population and another three contain 10% of the population each (.48). These are politically very different situations. The theory here suggests (see table in Appendix) that the former case will result in more conflictual equilibria than the latter. *Yet no empirical work that attempts* *to capture coalitional structures using the ELF measure alone, or any transformation of it, will be able to distinguish between these polities.*[14]

The theory suggests a fix for the problem of the ambiguity of ELF. It suggests (as evidenced by the figure above) that a better predictor for aggregate levels of violence than ELF and ELF-squared is given by ELF combined with either a variable that records the number of coalitions in each coalitional structure or a series of intercept terms for the number of groups in each coalitional structure. Such a specification should outperform a quadratic specification.

A test for this hypothesis is presented in the first three columns in Table 1 below. The table reports results from a Poisson AR(1) model of the incidence of guerilla wars in Africa 1975-1995.[15] The econometric model is extremely simplified: the number of guerilla wars is taken to be a measure of the impacts of bargaining failure across groups. This is assumed to be a function of the ethnolinguistic structure of the country (which in turn is assumed to correspond to coalitional structure) conditional upon a small number of structural and economic features of the country: size, wealth, urbanization, political institutions and growth rates. The models suggest that

1. ELF, entered as a quadratic, produces statistically significant coefficients (and these describe a parabola peaking at ELF=.46). But:

2. ELF, entered linearly alongside a simple count of the number of groups in the polity, is also statistically significant and correctly signed. And indeed such a model explains somewhat more of the variation in the dependent variable. Finally:

3. When both a quadratic term and a count term are entered, the quadratic term drops out while the coefficient on the count term is unaffected.

These results provide encouraging evidence for the model produced in this paper.[16] By developing a theoretically precise relationship between coalitional structure and political outcomes we have increased our

[14] For a more thorough and scathing account of the skeletons in ELF's closet, see Dan Posner (2000).

[15] The measure of guerilla warfare is taken from Arthur Banks' data archive. Banks defines guerilla wars as "any armed activity, sabotage, or bombings carried on by independent bands of citizens or irregular forces and aimed at the overthrow of the present regime." The list of countries that experienced one or more incidents of guerilla warfare per year is given in the final Appendix.

[16] We note however that while the predicted effects of the number of groups and the fractionalization of the polity are qualitatively consistent with the results from our model, the relative magnitudes of the effects differ somewhat from the formal

understanding of regularities already observed in African data. What is more, the theory has led us to alter the functional form used to study the data and improved our ability to explain observed variations.

Table I: Explaining the Incidence of Guerilla Wars in Africa (1975-1995)

Model:	I	II	III
Lag	0.46***	0.47***	0.46***
	(17.04)	(17.54)	(17.31)
ELF	3.60***	1.11***	2.22*
	(2.9)	(2.54)	(1.63)
ELF^2	-3.92***		-1.41
	(-3.05)		(-0.87)
Number of Ethnic Groups		-0.08***	-0.07***
		(-4.00)	(-2.73)
Executive Scales	-0.38***	-0.30***	-0.31***
	(-3.88)	(-3.06)	(-3.14)
Land Area	1.05E-08***	1.28E-08***	1.24E-08***
	(9.47)	(10.19)	(9.41)
Urbanization	-0.06***	-0.06***	-0.06***
	(-5.6)	(-5.5)	(-5.48)
Per Capita GDP	-3.78E-04*	-3.03E-04	-3.34E-04
	(-1.64)	(-1.36)	(-1.48)
Lag of GDP Growth Rate	-0.01	-0.01	-0.01
	(-1.2)	(-1.08)	(-1.12)
Constant	-0.76**	-0.67**	-0.77**
	(-2.01)	(-1.96)	(-2.12)
Number of Obs	831	831	831
R^2	0.3604	0.3772	0.3745
Adjusted R^2	0.3542	0.3712	0.3677

Notes: Dependent variables is the number of guerilla wars per country per year. Model is a Poisson AR(1), as described in Katsouyanni et al (1996) and Schwartz et al (1996). t-statistics are in parenthesis and asterisks denote significance at the 99% (***), 95% (**) and 90% (*) levels. Fitted values, variable definitions, sources and summary statistics are given in the Appendices.

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predictions. The econometric model suggests that countries with many equal sized groups engage in more conflictual action than polities with just two equal sized groups. To compare the relative effects of fragmentation of groups and number of groups, see the figure of fitted values printed in the Appendix.

4.7 Summary of Results from the Study of Coalitional Structures and an Example Illustrating Some of the Results

Summary of Results

Little Leviathans: If coalitions can negotiate effectively in a **TU** game, then the group of actors in a coalition acts *as if* it were a single actor whose ideal point is the simple average of the ideals of the members of the coalition and whose strength is the sum of the strengths of the members, multiplied by the number of members in a coalition.

Equilibrium. The equilibrium in the coalitional stage game is analogous to the equilibrium in the atomistic stage game: it is given by the weighted average of the ideals of the fictitious players. This is itself a weighted average of the ideals of individual players where the weights are a function of the size of the coalitions of which the players are a part and of the strengths of the other members of their coalition.

Unified Actions. All members of a coalition apply force in the same direction. The magnitude of force applied by each member of a coalition is increasing in his strength and decreasing in his weighting in the objective function of the coalition.

Efficiency Gains. In the **TU** game, equilibrium in the grand coalitional structure improves upon equilibrium under the atomistic structure in two distinct ways. First, the location of the stable point in the grand coalitional game (the unweighted utilitarian maximum) coupled with appropriate transfers Pareto dominates the stable point in the atomistic game (the weighted maximum without transfers). Second, in equilibrium the grand coalition expends no energy in attempting to alter the status quo.

Aggregate Levels of Violence in Equilibrium. The aggregate level of wasted efforts to force a change in the status quo is a function of the dispersion of the simple averages of the ideals of the players in the set of coalitions, and the sum of the strengths of the coalitions. Polarized coalitional structures that increase the dispersion across coalitions therefore increase aggregate levels of violence ceteris paribus. Coarser partitions that fail to reduce dispersion result in higher levels of violence, even though more bilateral bargains may be enforced relative to a finer partitioning.

I end this section with an example. This example illustrates key results from this section in a game with four players that may form into polarized or non-polarized coalitional structures.

Example

Consider a game with four players (a, b, c, d) of equal strength, $\pi^a = \pi^b = \pi^c = \pi^d = \frac{1}{4}$, and ideals (x^a, x^b, x^c, x^d) with $x^a = x^b$ and $x^c = x^d$. Assume the group divides into two coalitions of two members each. Assume first that a and b form coalition $\mathcal{C}^1$ while c and d form coalition $\mathcal{C}^2$. In this case $\mathcal{C} = \{\{a, b\}, \{c, d\}\}$ is a polarized coalitional structure. In this case, the equilibrium policy is given by $\frac{(x^{\mathcal{C}^1}+x^{\mathcal{C}^2})}{2} = \frac{(\frac{x^a+x^b}{2}+\frac{x^c+x^d}{2})}{2} = \frac{x^a+x^c}{2}$ and the total costs paid are give by: $\sum_{k=1}^{g} \pi^{\mathcal{C}^k} \rho(\mathcal{C})_{\mathcal{C}^k \in \mathcal{C}}$. Now recall that $\rho(\mathcal{C}) = \sum_{j=1}^{n} \frac{\sum_{k \in \mathcal{C}} \left[x_j^{\mathcal{C}^k} - \frac{x_j^{\mathcal{C}1}+x_j^{\mathcal{C}2}}{2} \right]^2}{|\mathcal{C}|}$ and so for this coalitional structure

$$\rho(\mathcal{C}) = \sum_{j=1}^{n} \frac{\left[\frac{x_j^a+x_j^b}{2} - \frac{x_j^a+x_j^c}{2}\right]^2 + \left[\frac{x_j^c+x_j^d}{2} - \frac{x_j^a+x_j^c}{2}\right]^2}{2} = \sum_{j=1}^{n} \left[\frac{x_j^a-x_j^c}{2}\right]^2.$$

Total costs will then be given by $2\sum_{j=1}^{n} \left[x_j^a - x_j^c\right]^2$. In essence, the groups will form into two unified coalitions each representable by a fictitious actor located at the players' shared ideal. However, if instead a and c formed coalition $\mathcal{D}^1$ while b and d form coalition $\mathcal{D}^2$, that is, $\mathcal{D} = \{\{a, c\}, \{b, d\}\}$. $\mathcal{D}$ is a non-polarized coalitional structure (there are "cross-cutting cleavages"). In this case the equilibrium policy would again be given by $\frac{(x^{\mathcal{D}^1}+x^{\mathcal{D}^2})}{2} = \frac{(\frac{x^a+x^c}{2}+\frac{x^b+x^d}{2})}{2} = \frac{x^a+x^c}{2}$. This time, however, since the fictional players share the same point, $\rho(\mathcal{D}) = \mathbf{0}$ for this game and hence total costs expended in equilibrium would be zero. This example illustrates the principal that if coalitions can make internal decisions costlessly, then players will do better in equilibrium if coalitions form among unlike types rather than, as we may expect, among likes. In this context cooperation between internally fragmented groups is preferable to conflict between homogenous groups. Note also that in this example $\rho(\mathcal{C}) = \rho(\mathcal{M})$. The coarser coalition $\mathcal{C}$ is as dispersed as $\mathcal{M}$. The total expenditure at the equilibrium point in the atomistic game is given then by $\sum_{j=1}^{n} \left[x_j^a - x_j^c\right]^2$ half that spent in the polarized game. In this case the fact that players in $\mathcal{C}^1$ and $\mathcal{C}^2$ internalize the effects of their actions upon the other member of their coalition results in them behaving more aggressively vis-à-vis members of the other coalition

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6 Appendices

6.1 Propositions and Proofs for the Repeat Play Game with Sophisticated Players

Lemma 1 (stated in text) *If a game with quadratic subutility functions and costs is iterated T times, where T is arbitrarily large, and all players are of equal strength, then the sum of all players' optimal actions, $\sum\limits_{i\in\mathcal{M}} v_t^{i*}$, in each period of play, $t = 1, 2, ..., T$, is given by a linear equation of the form: $\sum\limits_{i\in\mathcal{M}} v_t^{i*} = (\bar{x} - y_{t-1})k_t$, where $0 < k_t < 1$.*

Proof. It follows from **Lemma 2** and by induction that if Equations 24 and 25 (below) are satisfied in a game of length 1 then they hold for a game of arbitrary length. It is easy to verify that these conditions do hold in a game of length 1 or 2. Solving a game of length 2 the equations are satisfied by: $k_T = \frac{m\pi}{1+m\pi}$, $\alpha_T = \frac{m\pi^2}{1+m\pi}$, $\beta_T = \frac{\pi}{1+m\pi}$, $k_{T-1} = \frac{m\pi(1+m\pi)^2+m\pi(1+\pi)\delta}{(1+m\pi)^3+m\pi(1+\pi)\delta}$, $\alpha_{T-1} = \pi\left[\delta\frac{(1+\pi)}{(1+m\pi)} + \frac{m\pi(1+m\pi)^2+(m\pi-1)(1+\pi)\delta}{(1+m\pi)^3+m\pi(1+\pi)\delta}\right]$, and $\beta_{T-1} = \frac{\pi(1+m\pi)^2+\pi(1+\pi)\delta}{(1+m\pi)^3+m\pi(1+\pi)\delta}$ ■

Lemma 2. *If a game with quadratic subutility functions and costs is iterated for T periods (say from period 1 to period T), if all players are of equal strength, and if the following three conditions hold:*

(1) That for $t = 1, 2..., T$, a player's optimal action v_t^{i*} is given by some linear equation of the form:

$$v_t^{i*} = -\alpha_t\bar{x} - \beta_t y_{t-1} + (\alpha_t + \beta_t)x^i \tag{24}$$

where α_t and β_t are period-specific scalars and are constant over all players.

(2) That for $t = 1, 2..., T$, $k_t < 1$.

(3) That for $t = 1, 2..., T$, the sum of all players' optimal actions $\sum\limits_{i\in\mathcal{M}} v_t^{i*}$ is given by a linear equation of the form:

$$\sum_{i\in\mathcal{M}} v_t^{i*} = (\bar{x} - y_{t-1})k_t \tag{25}$$

where k_t is a period-specific scalar, constant over all players.

Then these three conditions will also hold for a game iterated for $T + 1$ periods for $t = 0, 1, ..., T$.

Proof. Before proceeding, note that since $y_1 = y_0 + \sum_{h\in\mathcal{M}} v_1^{h*}$ Equation 25 tells us that we may write $y_1 = k_1\bar{x} + (1-k_1)y_0$ and $y_2 = k_2\bar{x} + (1-k_2)y_1 = [1-(1-k_1)(1-k_2)]\,\bar{x} + [(1-k_1)(1-k_2)]\,y_0$. Proceeding in this manner and with repeated substitution for y_1, y_2... allows us to write more generally for $t > 0$:

$$y_t = [1 - \prod_{r=1}^{t}(1-k_r)]\bar{x} + [\prod_{r=1}^{t}(1-k_r)]y_0$$

Assume the conditions given in Equations 24 and 25 hold for a game of length T. Now consider a game of length $T+1$. In the first period of play of this game, each player maximizes, V_0^i,given by:

$$V_0^i = -|x^i - y_{-1} - \sum_{k\in\mathcal{M}} v_0^k|^2 - \frac{|v_0^i|^2}{\pi} - \sum_{t=1}^{T}\delta^t|x^i - y_t|^2 - \sum_{t=1}^{T}\delta^t\frac{|v_t|^2}{\pi}$$

Players using backwards induction, use Equations 24 and 25 to substitute for y_t and v_t. Using the fact that $y_0 = y_{-1} + \sum_{k\in\mathcal{M}} v_0^k$ we can now write the entire expression in terms of fixed quantities (including y_{-1}) and actions taken in time 0.

$$V_0^i = -|x^i - y_{-1} - \sum_{k\in\mathcal{M}} v_0^k|^2 - \frac{|v_0^i|^2}{\pi}$$
$$- \sum_{t=1}^{T}\delta^t|x^i - [1 - \prod_{r=1}^{t}(1-k_r)]\bar{x} - [\prod_{r=1}^{t}(1-k_r)][y_{-1} + \sum_{k\in\mathcal{M}} v_0^k]|^2$$
$$- \sum_{t=1}^{T}\delta^t\frac{|-\alpha_t\bar{x} - \beta_t[[1-\prod_{r=1}^{t-1}(1-k_r)]\bar{x} + [\prod_{r=1}^{t-1}(1-k_r)][y_{-1} + \sum_{h\in\mathcal{M}} v_0^h]] + (\alpha_t+\beta_t)x^i|^2}{\pi}$$

First order conditions for dimension j are given by:

$$v_{0,j}^i = \pi(x_j^i - y_{-1,j} - \sum_{k\in\mathcal{M}} v_{j,0}^k)$$
$$+ \sum_{t=1}^{T}\delta^t\pi\prod_{r=1}^{t}(1-k_r)\quad x^i - [1 - \prod_{r=1}^{t}(1-k_r)]\bar{x} - [\prod_{r=1}^{t}(1-k_r)][y_{-1} + \sum_{k\in\mathcal{M}} v_0^k]\Big]$$
$$+ \sum_{t=1}^{T}\delta^t\prod_{r=1}^{t-1}(1-k_r)[-\alpha_t\bar{x} - \beta_t[[1-\prod_{r=1}^{t-1}(1-k_r)]\bar{x} + [\prod_{r=1}^{t-1}(1-k_r)][y_{-1} +$$
$$\sum_{h\in\mathcal{M}} v_0^h]] + (\alpha_t+\beta_t)x^i]$$

Due to the excessive cumber of this expression it is useful now to define some constants as follows

$$\theta_1 \equiv \sum_{t=1}^{T}\delta^t\pi[\prod_{r=1}^{t}(1-k_r)]$$
$$\theta_2 \equiv \sum_{t=1}^{T}\delta^t\pi[\prod_{r=1}^{t}(1-k_r)][1 - \prod_{r=1}^{t}(1-k_r)]$$

$$\theta_3 \equiv \sum_{t=1}^{T} \delta^t \pi [\prod_{r=1}^{t} (1-k_r)][\prod_{r=1}^{t} (1-k_r)] = \theta_1 - \theta_2$$
$$\theta_4 \equiv \sum_{t=1}^{T} \delta^t [\prod_{r=1}^{t-1} (1-k_r)]\alpha_t$$
$$\theta_5 \equiv \sum_{t=1}^{T} \delta^t [\prod_{r=1}^{t-1} (1-k_r)]\beta_t$$
$$\theta_6 \equiv \sum_{t=1}^{T} \delta^t [\prod_{r=1}^{t-1} (1-k_r)][[1-\prod_{r=1}^{t-1} (1-k_r)]\beta_t$$
$$\theta_7 \equiv \sum_{t=1}^{T} \delta^t [\prod_{r=1}^{t-1} (1-k_r)][\prod_{r=1}^{t-1} (1-k_r)]\beta_t = \theta_5 - \theta_6$$

We may now write the first order conditions more simply, in vector form as:

$$v_0^i = (\pi+\theta_1+\theta_4+\theta_5)x^i - (\theta_2+\theta_4+\theta_6)\bar{x}_j - (\pi+\theta_3+\theta_7)[y_{-1} + \sum_{k\in\mathcal{M}} v_0^k] \quad (26)$$

Now, summing over the elements of $\mathcal{M}$ and replacing $\sum_{k\in\mathcal{M}} x^k$ with $m\bar{x}$ we have:

$$\sum_{k\in\mathcal{M}} v_0^k = m[\pi+\theta_3+\theta_7][\bar{x} - y_{-1} - \sum_{k\in\mathcal{M}} v_0^k] \quad (27)$$

We may then solve for $\sum_{k\in\mathcal{M}} v_0^k$:

$$\sum_{k\in\mathcal{M}} v_0^k = \frac{m[\pi+\theta_3+\theta_7]}{1+m[\pi+\theta_3+\theta_7]}[\bar{x} - y_{-1}] \quad (28)$$

Defining $k_0 = \frac{m[\pi+\theta_3+\theta_7]}{1+m[\pi+\theta_3+\theta_7]}$, Equation 28 establishes the first part of the claim. Furthermore, if $0 < k_t < 1$ for all $t \in \{1, 2, ...T\}$ then θ_3 and θ_7 are both positive and so $0 < k_0 < 1$. This establishes the second part of the proposition. We now return to consider v_0^i. Substituting for $\sum_{k\in\mathcal{M}} v_0^k$ from Equation 28 in Equation 26, we have:

$$v_0^i = (\pi+\theta_1+\theta_4+\theta_5)x^i - (\theta_2+\theta_4+\theta_6)\bar{x}_j - (\pi+\theta_3+\theta_7)[y_{-1} + k_0[\bar{x} - y_{-1}]] \quad (29)$$

Defining $\alpha_0 = (\theta_2+\theta_4+\theta_6+(\pi+\theta_3+\theta_7)k_0)$ and $\beta_0 = (\pi+\theta_3+\theta_7)(1-k_0)$ and noting that $\alpha_0+\beta_0 = (\pi+\theta_1+\theta_4+\theta_5)$ we may rearrange Equation 29 and substitute these terms to write:

$$v_0^i = -\alpha_0\bar{x} - \beta_0 y_0 + (\alpha_0+\beta_0)x^i \quad (30)$$

Equation 30 establishes the final part of the claim. ■

6.2 Table Relating Ethnolinguistic Fractionalization to Violence for 42 Coalitional Structures

The table printed in this Appendix shows the exhaustive set of 42 distinct coalitional structures for a game with ten players, each of identical strength. Each coalitional structure is represented as a particular partitioning of the population. For example, the partitioning $[\cdot\cdot\cdot\cdot|\cdot\cdot|\cdot|\cdot|\cdot|\cdot]$ represents a world in which there is one group containing 40% of the population, one containing 20% and four more containing 10% each. For each coalitional structure, the table reports a measure of fractionalization "ELF", given (for large groups) by ELF=$1-\sum_{\mathcal{C}^k\in\mathcal{C}}\left[\frac{|\mathcal{C}^k|}{|\mathcal{M}|}\right]^2$, where $|\mathcal{C}^k|$ is the number of individuals in coalition $\mathcal{C}^k$ from coalitional structure $\mathcal{C}$, and $|\mathcal{M}|$ denotes the total population of the polity. The table also reports the number of coalitions in the coalitional structure. Finally it reports the expected level of violence resulting in equilibrium from each coalitional structure. This is calculated as $\sum_{\mathcal{C}^k\in\mathcal{C}}\sum_{i\in\mathcal{C}^k}|\mathcal{C}^k|\pi^i\times\mathsf{E}\left[\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^\mathbf{k}})_{\mathcal{C}^k\in\mathcal{C}})\right]$ where $\mathsf{E}\left[\rho(\mathcal{C}|(\boldsymbol{\pi}^{\mathcal{C}^\mathbf{k}})_{\mathcal{C}^k\in\mathcal{C}})\right]$ denotes the expected (weighted) variance of the means of the ideal points of the coalitions when all members are independently drawn from some distribution with finite mean and variance.

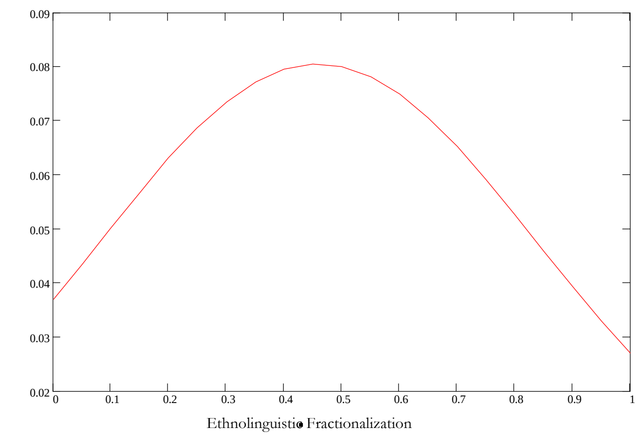
Table II: Coalitional Structure and Conflict

<table>
<tr><th>Coalitional Structure</th><th>ELF</th><th>#</th><th>Violence</th><th>Coalitional Structure</th><th>ELF</th><th>#</th><th>Violence</th></tr>
<tr><td>• • • • • | • • • • •</td><td>.5</td><td>2</td><td>.199</td><td>• • • • | • • • | • | • | •</td><td>.72</td><td>5</td><td>.11</td></tr>
<tr><td>• • • • • • | • • • •</td><td>.48</td><td>2</td><td>.187</td><td>• • • | • • | • • | • | • | •</td><td>.8</td><td>6</td><td>.106</td></tr>
<tr><td>• • • • | • • • | • • •</td><td>.66</td><td>3</td><td>.185</td><td>• • • | • • • | • | • | • | •</td><td>.78</td><td>6</td><td>.102</td></tr>
<tr><td>• • • • | • • • • | • •</td><td>.64</td><td>3</td><td>.176</td><td>• • | • • | • • | • | • | • | •</td><td>.84</td><td>7</td><td>.1</td></tr>
<tr><td>• • • • • | • • • | • •</td><td>.62</td><td>3</td><td>.159</td><td>• • • • • • • | • • | •</td><td>.46</td><td>3</td><td>.098</td></tr>
<tr><td>• • • • • | • • • • | •</td><td>.58</td><td>3</td><td>.155</td><td>• • • • | • • | • | • | • | •</td><td>.76</td><td>6</td><td>.096</td></tr>
<tr><td>• • • • • • • | • • •</td><td>.42</td><td>2</td><td>.153</td><td>• • • • • | • • | • | • | •</td><td>.68</td><td>5</td><td>.096</td></tr>
<tr><td>• • • | • • • | • • | • •</td><td>.74</td><td>4</td><td>.153</td><td>• • • • • • | • • | • | •</td><td>.58</td><td>4</td><td>.096</td></tr>
<tr><td>• • • | • • • | • • • | •</td><td>.72</td><td>4</td><td>.147</td><td>• • • • • • • • | • •</td><td>.32</td><td>2</td><td>.096</td></tr>
<tr><td>• • • • | • • | • • | • •</td><td>.72</td><td>4</td><td>.143</td><td>• • • | • • | • | • | • | • | •</td><td>.82</td><td>7</td><td>.092</td></tr>
<tr><td>• • • • | • • • | • • | •</td><td>.7</td><td>4</td><td>.137</td><td>• • | • • | • | • | • | • | • | •</td><td>.86</td><td>8</td><td>.089</td></tr>
<tr><td>• • • • • • | • • | • •</td><td>.54</td><td>3</td><td>.135</td><td>• • • | • | • | • | • | • | • | •</td><td>.84</td><td>8</td><td>.082</td></tr>
<tr><td>• • | • • | • • | • • | • •</td><td>.8</td><td>5</td><td>.133</td><td>• • • • | • | • | • | • | • | •</td><td>.78</td><td>7</td><td>.082</td></tr>
<tr><td>• • • • • • | • • • | •</td><td>.56</td><td>3</td><td>.132</td><td>• • | • | • | • | • | • | • | • | •</td><td>.88</td><td>9</td><td>.08</td></tr>
<tr><td>• • • | • • | • • | • • | •</td><td>.78</td><td>5</td><td>.128</td><td>• • • • • | • | • | • | • | •</td><td>.7</td><td>6</td><td>.077</td></tr>
<tr><td>• • • • | • • • • | • | •</td><td>.66</td><td>4</td><td>.127</td><td>• | • | • | • | • | • | • | • | • | •</td><td>.9</td><td>10</td><td>.085</td></tr>
<tr><td>• • • | • • • | • • | • | •</td><td>.76</td><td>5</td><td>.122</td><td>• • • • • • | • | • | • | •</td><td>.6</td><td>5</td><td>.075</td></tr>
<tr><td>• • • • • | • • | • • | •</td><td>.66</td><td>4</td><td>.12</td><td>• • • • • • • | • | • | •</td><td>.48</td><td>.4</td><td>.07</td></tr>
<tr><td>• • • • • | • • • | • | •</td><td>.64</td><td>4</td><td>.12</td><td>• • • • • • • • | • | •</td><td>.34</td><td>3</td><td>.062</td></tr>
<tr><td>• • • • | • • | • • | • | •</td><td>.74</td><td>4</td><td>.153</td><td>• • • • • • • • • | •</td><td>.18</td><td>2</td><td>.045</td></tr>
<tr><td>• • | • • | • • | • • | • | •</td><td>.82</td><td>6</td><td>.112</td><td>• • • • • • • • • •</td><td>0</td><td>1</td><td>0</td></tr>
</table>

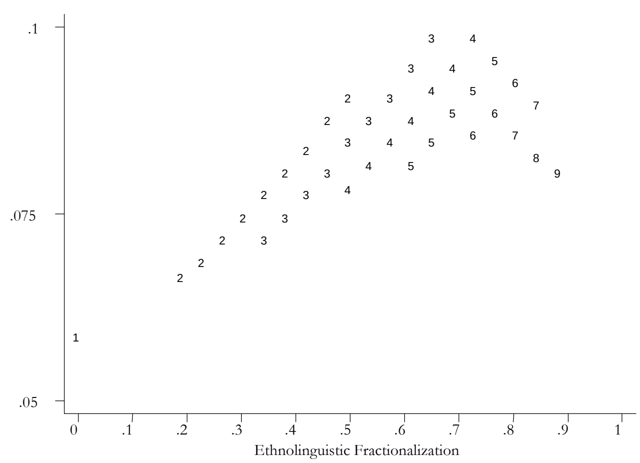
6.3 Fitted Values for Econometric Models 1 and 2.

The two figures in this appendix show fitted values for Poisson Models I and II, reported in Table 1 in the text. The fitted values show the expected number of guerilla wars in a country as a function of the ethnolinguistic structure of the country, all other control variables held at their means.

Model I (Quadratic)



Model II (Linear with Intercept)



This figure reports the fitted values for the expected number of guerilla wars (on the vertical axis) as a function of the level of ELF (on the horizontal axis) and the number of ethnic groups in a country (labels on the datapoints).

6.4 Variables Used to Produce Table 1

Description and Source

Variable	Description	Source
Guerilla Wars	Any armed activity, sabotage, or bombings carried on by independent bands of citizens or irregular forces and aimed at the overthrow of the present regime.	Banks (2000)
ELF	Ethnolinguistic Fractionalization	Soviet Union (1964)
Number of Ethnic Groups	Number of Ethnic Groups per Country	Soviet Union (1964)
Land Area	Land Area in Hectares	World Bank(2000)
Urbanization	% of Population Living in Urban Areas	World Bank. (2000)
GDP per capita	Real per capita PPP GDP chain index	Summers and Heston (1996)
Lag of GDP per capita growth		Summers and Heston (1996)
Executive Scales	Describes the competitiveness of the electoral system.	Ferree and Singh (2000)

Summary Statistics

Variable	Mean	s.d.	Min.	Max.
Guerilla Wars	.16	.46	0	5
ELF	.62	.28	0	.92
Number of Groups	9.33	5.51	1	26
Land Area	4.8e+07	5.6e+07	45000	2.4e+08
Urbanization	23.3	13.5	2	82
GDP per capita	1101	892	261	6965
Lag of GDP per capita growth	.58	7.93	-64.24	51.30
Executive Scales	3.3	1.5	1	6

Details on the Dependent Variable

According to the Banks dataset, the following country years experienced one or more incidents of guerilla warfare: Angola 75-79, 81-90, 93-95; Burundi 72, 73; Chad 70-72, 74, 78, 80, 82-89; Ivory Coast 70; Djibouti 93,94, 95; Ethiopia 70-71, 74-78, 81-92; Guinea 70; Lesotho 70, 81-83; Liberia 91-95; Madagascar 91; Mauritania 76-79; Mozambique 75-85, 88-91; Namibia 81-85, 87-88; Namibia 82-85, 87-88; Nigeria 70, 94; Rwanda 91; Sierra Leone 95; Somalia 81-82, 88-95; Sudan 70-72, 84-86, 89-95; Swaziland 84-85; Uganda 72, 81-83, 85-89, 91; Zaire, 70, 77, 88-89 ; Zimbabwe 70, 72-81, 83-85.